# Table of Non-Linear Simultaneous Equations

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**Abstract:** ***Solving non-linear simultaneous equations is not so straightforward as its linear counterpart and is usually a challenge especially for the mathematically average person. The existing methods involve the use of substitution, or in some cases, elimination. This paper introduces a new method of solving various forms of non-linear simultaneous equations via Cramer's Rule whereby the task is reduced to simple calculations thatrequires nothing more than basic mathematics of addition, subtraction, multiplication, division, square and squareroot of some real numbers, hence making it possible for a person who doesn't know the substitution procedure at all to be able to determine the correct answers to the said equations. A total of 65 different general forms of non-linear simultaneous equations together with their solutions in closed forms are tabulated in the proposed Table of Non-linear Simultaneous Equations. Hence, for a given set of non-linear simultaneous equations in two variables that matches one of the tabulated forms, one just read off the solutions from the table, plug-in the corresponding constants and perform some simple calculations to obtain the answers. Furthermore, since the computation involve nothing more than straightforward arithmetic, it can be easily programed using basic formula in Excel spreadsheet. One just need to enter the few constants from the two equations into the spreadsheet, the answers will be obtained in a second.***

**Keywords:** Non-linear equations; simultaneous equations; Cramer's rule

## 1. Introduction

It is essential to model some real-world systems with some mathematical equations in order to have a better insight to the systems and hence able to control, forecast future outputs, prevent undesirable situations and understand various phenomena related to the systems. With modeling, we can easily change some settings and observe its effects and interpret the system behavior without actually disturbing the physical system.

Numerous disciplines, both technical and non-technical, employ mathematical modeling extensively. These variety of fields include engineering, pure sciences, social sciences, medicine and economics.

Most practical systems are non-linear in nature and often require more than one equations for a full representation. If the situation allows, these non-linear systems may be approximated as linear systems and thus greatly simplifies the solutions to the equations since there are plenty of methods available for solving linear simultaneous equations. However, errors are bound to be introduced with this linearization.

In modeling real-world systems, we often face with the task of solving non-linear simultaneous equations. Some of these can be done using readily available methods such as substitution and elimination. However, this solution process, though simple, may be tedious especially if we have to repeat the computation with many different sets of constants and if the constants involved have many decimal places. Hence, reducing the procedure of solving non-linear simultaneous equations to merely performing straightforward arithmetic and could be done by persons with minimal mathematical capability is highly desirable. Moreover, instantly obtaining the correct answers and do a verification using pre-programed spreadsheet is time-saving and error free.

## 2. Derivation of Results

First of all, let's define the following quantities where $a, b, c, d, m$ and $n$ are the six constants in the two independent simultaneous non-linear equations in two variables. These definitions will be used throughout this paper.

$$\alpha = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad ; \; A = ad \;\; , \;\; B = bc$$

$$\beta = \begin{vmatrix} a & m \\ c & n \end{vmatrix} = an - cm \quad ; \; C = an \;\; , \;\; D = cm$$

$$\gamma = -\begin{vmatrix} b & m \\ d & n \end{vmatrix} = dm - bn; \; E = dm \;\; , \;\; F = bn$$

The various different types of non-linear simultaneous equations in two variables are classified into five general forms. This paper demonstrates the derivation of solution for one type per general form because the derivation for other types are similar.

### 2.1 Forms Involving $x^2, \frac{1}{x}, x^p$ and $\frac{1}{x^p}, p = 1, 2, 3, \cdots$

Now, let's start with the system below, where $p$ is a positive integer.

$$ax^2 + bx^p y = m \qquad (2.1.1)$$

$$cx + dx^p y = n \qquad (2.1.2)$$

This system can be written in matrix form as follows:

$$\begin{bmatrix} ax & bx^p \\ c & dx^p \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} \qquad (2.1.3)$$

$$|A| = \begin{vmatrix} ax & bx^p \\ c & dx^p \end{vmatrix} = adx^{p+1} - bcx^p = (adx - bc)x^p = (Ax - B)x^p \qquad (2.1.4)$$

$$|A_x| = \begin{vmatrix} m & bx^p \\ n & dx^p \end{vmatrix} = dmx^p - bnx^p = (dm - bn)x^p = \gamma x^p \qquad (2.1.5)$$

$$|A_y| = \begin{vmatrix} ax & m \\ c & n \end{vmatrix} = anx - cm = Cx - D \qquad (2.1.6)$$

Using Cramer's Rule, the two solutions are

$$x = \frac{\gamma x^p}{(Ax-B)x^p} = \frac{\gamma}{(Ax-B)} \qquad (2.1.7)$$

$$\text{And } \; y = \frac{Cx - D}{(Ax-B)x^p}. \qquad (2.1.8)$$

Rearranging equation (2.1.7), we obtained the quadratic equation

$$Ax^2 - Bx - \gamma = 0 \qquad (2.1.9)$$

which solutions are given by

$$x_{1,2} = \frac{B \pm \sqrt{B^2 + 4A\gamma}}{2A}. \qquad (2.1.10)$$

These two solutions $x_1$ and $x_2$ are then substituted into equation (2.1.8) to obtain the corresponding $y_1$ and $y_2$.

### 2.2 Numerical Example 1

Given the following system of non-linear simultaneous equations

$$6x^2 + 4x^3y = 11$$
$$-5x + 7x^3y = 2.$$

By comparing to equations (2.1.1) and (2.1.2), we see that $a = 6, b = 4, c = -5, d = 7, m = 11, n = 2$ and $p = 3$. We can easily obtain $A = ad = 42, B = bc = -20, C = an = 12, D = cm = -55$ and $\gamma = dm - bn = 69$. We next substitute these values into equation (2.1.10):

$$x_{1,2} = \frac{B \pm \sqrt{B^2 + 4A\gamma}}{2A} = \frac{-20 \pm \sqrt{(-20)^2 + 4(42)(69)}}{2(42)} = \frac{-20 \pm 109.51}{84}$$

$$x_1 = 1.07 \quad ; \quad x_2 = -1.54$$

Now, substitute these two $x$ values into equation (2.1.8) to obtain:

$$y_1 = \frac{(12)(1.07) - (-55)}{[(42)(1.07) - (-20)](1.07)^3} = 0.85 \quad ; \quad y_2 = \frac{(12)(-1.54) - (-55)}{[(42)(-1.54) - (-20)](-1.54)^3} = 0.22$$

It could be easily verified that these two set of solutions do satisfy the given system.

Note 1: It seems that quite a bit of calculations, though straightforward, are required in using the derived formula. However, these formula can be easily programed into Excel spreadsheet. All we have to do is to enter the seven constants from the system into the spreadsheet and the answer will be obtained instantly as shown in Figure 1 below.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | $ax^2 + bx^py = m$ | | | | | |
| | | $cx + dx^py = n$ | | | | p = | 3 |
| a = | 6 | | b = | 4 | | m = | 11 |
| c = | -5 | | d = | 7 | | n = | 2 |
| | | *x1* = | 1.07 | | *y1* = | 0.87 | |
| | | *x2* = | -1.54 | | *y2* = | 0.22 | |

**Figure 1:** Spreadsheet for Numerical Example 1

Note 2: This same system can be solved using the Elimination method. However, if required, the whole elimination process must be repeated for different sets of constants. Using this spreadsheet, we could solve dozens of simultaneous equations of this form in just a minute or two.

Note 3: The answers obtained may be effortlessly verified by substituting them back into the given equations using the same pre-programed spreadsheet.

### 2.2 Forms involving $\sqrt{x}$, $x > 0$

We now consider the system

$$a\sqrt{x} + by = m \qquad (2.2.1)$$
$$cx + d\sqrt{x}y = n\,. \qquad (2.2.2)$$

The equivalent equation in matrix form is

$$\begin{bmatrix} \frac{a}{\sqrt{x}} & b \\ c & d\sqrt{x} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} \quad . \qquad (2.2.3)$$

$$|A| = \begin{vmatrix} \frac{a}{\sqrt{x}} & b \\ c & d\sqrt{x} \end{vmatrix} = ad - bc = \alpha \qquad (2.2.4)$$

$$|A_x| = \begin{vmatrix} m & b \\ n & d\sqrt{x} \end{vmatrix} = dm\sqrt{x} - bn = E\sqrt{x} - F \qquad (2.2.5)$$

$$|A_y| = \begin{vmatrix} \frac{a}{\sqrt{x}} & m \\ c & n \end{vmatrix} = \frac{an}{\sqrt{x}} - cm = \frac{C - D\sqrt{x}}{\sqrt{x}} \qquad (2.2.6)$$

Employing Cramer's Rule, we obtained

$$x = \frac{E\sqrt{x} - F}{\alpha} \qquad (2.2.7)$$

and

$$y = \frac{C - D\sqrt{x}}{\alpha\sqrt{x}}\,. \qquad (2.2.8)$$

Rearranging equation (2.27) yields

$$\alpha x - E\sqrt{x} + F = 0\,. \qquad (2.2.9)$$

If we let $u = \sqrt{x}$and$u^2 = x$, then we obtain the quadratic equation

$$\alpha u^2 - Eu + F = 0 \qquad (2.2.10)$$

which has the solutions

$$u_{1,2} = \frac{E \pm \sqrt{E^2 - 4\alpha F}}{2\alpha}\,. \qquad (2.2.11)$$

Since $x = u^2$, we have

$$x_{1,2} = \left(\frac{E \pm \sqrt{E^2 - 4\alpha F}}{2\alpha}\right)^2. \qquad (2.2.12)$$

### 2.2.1 Numerical Example 2

Let's say the system at hand is given by

$11\sqrt{x} + 12y = 28$

$7x - 13\sqrt{x}y = 22$ .

The constants are $a = 11, b = 12, c = 7, d = -13, m = 28, n = 22$ and the parameters are calculated to be $C = 242, D = 196, E = -364, F = 264$ and $\alpha = -227$. The values for $x$ and $y$ are:

$$x_{1,2} = \left(\frac{-364 \pm \sqrt{(-364)^2 - 4(-227)(264)}}{2(-227)}\right)^2$$

$$= \left(\frac{-364 \pm 610.09}{-454}\right)^2$$

$$x_1 = 0.294$$

$$x_2 = 4.603$$

$$y_1 = \frac{(242)-(196)\sqrt{0.294}}{(-227)\sqrt{0.294}} = -1.103$$

$$y_2 = \frac{(242)-(196)\sqrt{4.603}}{(-227)\sqrt{4.603}} = 0.367$$

Figure 2 shows the same results from the spreadsheet. However, from the verification box which was programed to substitute the $x$ and $y$values back into the given equations, we see that only ($x_2$,$y_2$) satisfy both the equations. In this case, the answer ($x_1$,$y_1$) must be discarded and this is not unusual in algebra.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a = | 11 | | b = | 12 | | m = | 28 | | α = | -227 | A = | -143 | D = | 196 |
| c = | 7 | | d = | -13 | | n = | 22 | | β = | 46 | B = | 84 | E = | -364 |
| | | *x1* = | 0.294 | | *y1* = | -1.103 | | | γ = | -628 | C = | 242 | F = | 264 |
| | | *x2* = | 4.603 | | *y2* = | 0.367 | | | | | | | | |

| Verification: | | | | | | | |
|---|---|---|---|---|---|---|---|
| (x1, y1) | Eq 1: | LHS = | -7.278 | (x2, y2) | Eq 1: | LHS = | 28 |
| | Eq 2: | LHS = | 9.8314 | | Eq 2: | LHS = | 22 |

**Figure 2:** Spreadsheet for Numerical Example 2

### 2.3 Forms involving $e^{kx}, \ln(qx)$ , $x^p$and $\frac{1}{x^p}$ , $k, p = 1, 2, 3, \cdots$

Next, let's consider the system of the following form.

$$ae^{kx} + bx^p(\ln qx)\, y = m \qquad (2.3.1)$$

$$ce^{kx} + dx^p(\ln qx)y = n \qquad (2.3.2)$$

$$\begin{bmatrix} a\frac{e^{kx}}{x} & bx^p(\ln qx) \\ c\frac{e^{kx}}{x} & dx^p(\ln qx) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} \qquad (2.3.3)$$

$$|A| = \begin{vmatrix} a\frac{e^{kx}}{x} & bx^p(\ln qx) \\ c\frac{e^{kx}}{x} & dx^p(\ln qx) \end{vmatrix} = ad\frac{x^p e^{kx}\ln qx}{x} - bc\frac{x^p e^{kx}\ln qx}{x} = \alpha\left(\frac{x^p e^{kx}\ln qx}{x}\right)$$

(2.3.4)

$$|A_x| = \begin{vmatrix} m & bx^p(\ln qx) \\ n & dx^p(\ln qx) \end{vmatrix} = dmx^p(\ln qx) - bnx^p(\ln qx) = \gamma x^p(\ln qx) \qquad (2.3.5)$$

$$|A_y| = \begin{vmatrix} a\frac{e^{kx}}{x} & m \\ c\frac{e^{kx}}{x} & n \end{vmatrix} = an\frac{e^{kx}}{x} - cm\frac{e^{kx}}{x} = \beta\frac{e^{kx}}{x} \qquad (2.3.6)$$

$$x = \frac{\gamma x^p(\ln qx)x}{\alpha x^p e^{kx}\ln qx} = \frac{\gamma x}{\alpha e^{kx}} \qquad (2.3.7)$$

$$e^{kx} = \frac{\gamma}{\alpha} \qquad (2.3.8)$$

$$x = \frac{1}{k}\ln\left(\frac{\gamma}{\alpha}\right) \;,\; \frac{\gamma}{\alpha} > 0 \;. \qquad (2.3.9)$$

$$y = \frac{\beta e^{kx}}{x} \cdot \frac{x}{\alpha x^p e^{kx}\ln qx} \qquad (2.3.10)$$

$$y = \frac{\beta}{\alpha x^p \ln(qx)} \;,\; qx > 0 \qquad (2.3.11)$$

### 2.3.1 Numerical Example 3

$$2e^{3x} - 2x^4(\ln 2x)y = 6$$

$$3e^{3x} + 8x^4(\ln 2x)y = 7$$

$$a = 2, b = -2, c = 3, d = 8, m = 6, n = 7, k = 3, p = 4 \text{ and } q = 2.$$

$$\alpha = 22, \beta = -4 \text{ and } \gamma = 62.$$

Using the derived formula (2.3.9) and (2.3.11), we obtain

$$x = \frac{1}{3}\ln\left(\frac{62}{22}\right) = 0.345 \qquad \text{and}$$

$$y = \frac{-4}{(22)(0.345)^4\ln(2\times 0.345)} = 34.587 \;.$$

These two solutions are the same as those obtained using the spreadsheet shown in Figure 3 with a little round-off error. The verification of the answers is done automatically by the spreadsheet.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $ae^{kx} + bx^p(\ln qx)\, y = m$ | | | | | k = | 3 | | | | | | | |
| $ce^{kx} + dx^p(\ln qx)y = n$ | | | | | p = | 4 | | | | | | | |
| | | | | | q = | 2 | | | | | | | |
| a = | 2 | | b = | -2 | m = | 6 | | α = | 22 | A = | 16 | D = | 18 |
| c = | 3 | | d = | 8 | n = | 7 | | β = | -4 | B = | -6 | E = | 48 |
| | | *x* = | 0.345 | *y* = | 34.540 | | | γ = | 62 | C = | 14 | F = | -14 |

| Verification: | | | |
|---|---|---|---|
| (x1, y1) | Eq 1: | LHS = | 6 |
| | Eq 2: | LHS = | 7 |

**Figure 3:** Spreadsheet for Numerical Example 3

### 2.4 Forms involving Trigonometric Functions

Now, Let's derive the formula for forms involving trigonometric functions. Consider the following system:

$$a\sin kx + bx^p y = m \qquad k,p = 1,2,\cdots \qquad (2.4.1)$$

$$c\sin kx + dx^p y = n \qquad (2.4.2)$$

$$\begin{bmatrix} a\frac{\sin kx}{x} & bx^p \\ c\frac{\sin kx}{x} & dx^p \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix} \qquad (2.4.3)$$

$$|A| = \begin{vmatrix} a\frac{\sin kx}{x} & bx^p \\ c\frac{\sin kx}{x} & dx^p \end{vmatrix} = adx^{p-1}\sin kx - bcx^{p-1}\sin kx = \alpha x^{p-1}\sin kx \quad (2.4.4)$$

$$|A_x| = \begin{vmatrix} m & bx^p \\ n & dx^p \end{vmatrix} = dmx^p - bnx^p = \gamma x^p \quad (2.4.5)$$

$$|A_y| = \begin{vmatrix} a\frac{\sin kx}{x} & m \\ c\frac{\sin kx}{x} & n \end{vmatrix} = an\frac{\sin kx}{x} - cm\frac{\sin kx}{x} = \frac{\beta\sin kx}{x} \quad (2.4.6)$$

$$x = \frac{\gamma x^p}{\alpha x^{p-1}\sin kx} = \frac{\gamma x}{\alpha\sin kx} \qquad (2.4.7)$$

$$\sin kx = \frac{\gamma}{\alpha} \qquad (2.4.8)$$

$$x = \frac{1}{k}\sin^{-1}\left(\frac{\gamma}{\alpha}\right) \qquad (2.4.9)$$

Depends on the value of $x$ in radian, the angle $kx$could be in $Q1$, $Q2$, $Q3$ or $Q4$. Keeping in mind that these formulae are intended for those who do not know or do not want to solve the simultaneous non-linear equations in the conventional ways, the solutionfor $x$ is generalize as

$$x = n\pi \pm \frac{1}{k}\sin^{-1}\left(\frac{\gamma}{\alpha}\right), \quad n = 0,1 \ ; \ -1 < \frac{\gamma}{\alpha} < 1 \quad (2.4.10)$$

which will yield four different values for $x$. We could choose the correct one, two or all four, by substituting them back into the given equations which could be done effortlessly with the aid of spreadsheet.

The value for $y$ is given by

$$y = \frac{\beta\sin kx}{x\alpha x^{p-1}\sin kx} = \frac{\beta}{\alpha x^p}. \qquad (2.4.11)$$

#### 2.4.1 Numerical Example 4

$$2\sin 3x - 4x^2 y = -9$$

$$3\sin 3x + x^2 y = 5$$

$a = 2, b = -4, c = 3, d = 1, m = -9, n = 5, k = 3, p = 2,\ \alpha = 14, \beta = 37, \gamma = 11.$

Using the derived formula (2.4.10), the $x$ values are calculated to be:

$$x_1 = \frac{1}{3}\sin^{-1}\left(\frac{11}{14}\right) = 0.30 \qquad ;$$

$$x_2 = -\frac{1}{3}\sin^{-1}\left(\frac{11}{14}\right) = -0.30$$

$$x_3 = \pi + \frac{1}{3}\sin^{-1}\left(\frac{11}{14}\right) = 3.44 \qquad ;$$

$$x_4 = \pi - \frac{1}{3}\sin^{-1}\left(\frac{11}{14}\right) = 2.84$$

The corresponding values for $y$from (2.4.11) are:

$$y_1 = \frac{37}{(14)(0.30)^2} = 29.37 \qquad ;$$

$$y_2 = \frac{37}{(14)(-0.30)^2} = 29.37$$

$$y_3 = \frac{37}{(14)(3.44)^2} = 0.22 \qquad ;$$

$$y_4 = \frac{37}{(14)(2.84)^2} = 0.33$$

One can't just accept these four sets of solutions but should select the correct ones by putting them back into the given equations. It is found that only the two pairs $(x_1, y_1)$and $(x_4, y_4)$ satisfy the system. Hence, we have to reject $(x_2, y_2)$and$(x_3, y_3)$.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | k = | 3 | | | | | | | |
| | | | | | | p = | 2 | | | | | | | |
| a = | 2 | | b = | -4 | | m = | -9 | | α = | 14 | A = | 2 | D = | -27 |
| c = | 3 | | d = | 1 | | n = | 5 | | β = | 37 | B = | -12 | E = | -9 |
| | | *x1* = | 0.30 | | *y1* = | 29.12 | | | γ = | 11 | C = | 10 | F = | -20 |
| | | *x2* = | -0.30 | | *y2* = | 29.12 | | | Verification: | | Eq 1 | Eq 2 | | |
| | | *x3* = | 3.44 | | *y3* = | 0.22 | | | *(x1,y1)* | LHS = | -9 | 5 | | |
| | | *x4* = | 2.84 | | *y4* = | 0.33 | | | *(x2,y2)* | LHS = | -12.1429 | 0.28571 | | |
| | | | | | | | | | *(x3,y3)* | LHS = | -12.1429 | 0.28571 | | |
| | | | | | | | | | *(x4,y4)* | LHS = | -9 | 5 | | |

**Figure 4:** Spreadsheet for Numerical Example 4

### 2.5 Forms involving reciprocal of Trigonometric Function

Reciprocal of trigonometric functions refers to cosecant, secant and cotangent. The derivation of the solutions for these functions are very similar to the trigonometric functions and hence are omitted.

### Table of Non-linear Simultaneous Equations

These results are tabulated in five general formsfor easy reference.Each derived formula was tested with several numerical examples using Excel spreadsheet.

| *Forms involving* $x^2, \frac{1}{x}, x^p$ *and* $\frac{1}{x^p}, p = 1, 2, 3, \cdots$ | | |
|---|---|---|
| 1 | $ax^2 + by = m$<br>$cx + dy = n$ | $x_{1,2} = \frac{B \pm \sqrt{B^2 + 4A\gamma}}{2A}$<br>$y = \frac{Cx - D}{Ax - B}$ |
| 2 | $ax^2 + bx^p y = m$<br>$cx + dx^p y = n$ | $x_{1,2} = \frac{B \pm \sqrt{B^2 + 4A\gamma}}{2A}$<br>$y = \frac{Cx - D}{(Ax - B)x^p}$ |
| 3 | $ax^2 + \frac{b}{x^p} y = m$<br>$cx + \frac{d}{x^p} y = n$ | $x_{1,2} = \frac{B \pm \sqrt{B^2 + 4A\gamma}}{2A}$<br>$y = \frac{(Cx - D)x^p}{Ax - B}$ |
| 4 | $ax^2 + by = m$<br>$cx^2 + dy = n$ | $x = \pm\sqrt{\frac{\gamma}{\alpha}}$<br>$y = \frac{\beta}{\alpha}$ |
| 5 | $ax^2 + bx^p y = m$<br>$cx^2 + dx^p y = n$ | $x = \pm\sqrt{\frac{\gamma}{\alpha}}$<br>$y = \frac{\beta}{\alpha x^p}$ |
| 6 | $ax^2 + \frac{b}{x^p} y = m$<br>$cx^2 + \frac{d}{x^p} y = n$ | $x = \pm\sqrt{\frac{\gamma}{\alpha}}$<br>$y = \frac{\beta x^p}{\alpha}$ |
| 7 | $a\frac{1}{x} + by = m$<br>$cx + dy = n$ | $x_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 + 4AB}}{2B}$<br>$y = \frac{C - Dx^2}{A - Bx^2}$ |
| 8 | $a\frac{1}{x^2} + by = m$<br>$c\frac{1}{x} + dy = n$ | $x_{1,2} = \frac{-B \pm \sqrt{B^2 + 4A\gamma}}{2\gamma}$<br>$y = \frac{C - Dx}{A - Bx}$ |
| 9 | $a\frac{1}{x^2} + by = m$<br>$cx + dxy = n$ | $x_{1,2} = \frac{F \pm \sqrt{F^2 + 4A(E + B)}}{2(E + B)}$<br>$y = \frac{(C - Dx^3)}{Ax - Bx^3}$ |
| 10 | $a\frac{1}{x^2} + by = m$<br>$c\frac{1}{x^2} + dy = n$ | $x = \pm\sqrt[2]{\frac{\alpha}{\gamma}}$<br>$y = \frac{\beta}{\alpha}$ |
| 11 | $a\frac{1}{x^p} + by = m$<br>$c\frac{1}{x^p} + dy = n$ | $x = \pm\sqrt[p]{\frac{\alpha}{\gamma}}$<br>$y = \frac{\beta}{\alpha}$ |
| 12 | $a\frac{1}{x^2} + bxy = m$<br>$c\frac{1}{x^2} + dxy = n$ | $x = \pm\sqrt[2]{\frac{\gamma}{\alpha}}$<br>$y = \frac{\beta}{\alpha x}$ |
| 13 | $a\frac{1}{x^2} + bx^p y = m$<br>$c\frac{1}{x^2} + dx^p y = n$ | $x = \pm\sqrt[2]{\frac{\gamma}{\alpha}}$<br>$y = \frac{\beta}{\alpha x^p}$ |
| 14 | $ax^2 + by^2 = m$<br>$cx + dy^2 = n$ | $x_{1,2} = \frac{B \pm \sqrt{B^2 + 4A\gamma}}{2A}$ |

| | | |
|---|---|---|
| | | $y = \pm\sqrt{\dfrac{Cx - D}{Ax - B}}$ |
| *Forms involving* $\sqrt{x}$ | | |
| 15 | $a\sqrt{x} + by = m$<br>$cx + by = m$ | $x_{1,2} = \left(\dfrac{A \pm \sqrt{A^2 - 4B\gamma}}{2B}\right)^2$<br>$y = \dfrac{C - D\sqrt{x}}{A - B\sqrt{x}}$ |
| 16 | $a\sqrt{x} + by = m$<br>$c\sqrt{x} + dy = n$ | $x = \left(\dfrac{\gamma}{\alpha}\right)^2$<br>$y = \dfrac{\beta}{\alpha}$ |
| 17 | $a\sqrt{x} + by = m$<br>$cx + d\sqrt{x}y = n$ | $x_{1,2} = \left(\dfrac{E \pm \sqrt{E^2 - 4\alpha F}}{2\alpha}\right)^2$<br>$y = \dfrac{C - D\sqrt{x}}{\alpha\sqrt{x}}$ |
| 18 | $a\sqrt{x} + by = m$<br>$c\sqrt{x} + d\sqrt{x}y = n$ | $x = \left(\dfrac{(B+E) \pm \sqrt{(B+E)^2 - 4AF}}{2A}\right)^2$<br>$y = \dfrac{C - D}{A\sqrt{x} - B}$ |
| 19 | $a\sqrt{x} + b\sqrt{x}y = m$<br>$cx + d\sqrt{x}y = n$ | $x = \left(\dfrac{A \pm \sqrt{A^2 - 4B\gamma}}{2B}\right)^2$<br>$y = \dfrac{C - D\sqrt{x}}{A\sqrt{x} - Bx}$ |
| 20 | $a\sqrt{x} + bxy = m$<br>$cx + dxy = n$ | $x = \left(\dfrac{A \pm \sqrt{A^2 - 4B\gamma}}{2B}\right)^2$<br>$y = \dfrac{C - D\sqrt{x}}{Ax - B\sqrt{x^3}}$ |
| 21 | $a\sqrt{x} + bxy = m$<br>$c\sqrt{x} + dxy = n$ | $x = \left(\dfrac{\gamma}{\alpha}\right)^2$<br>$y = \dfrac{\beta}{\alpha x}$ |
| *Forms involving* $e^{kx}, \ln(qx), x^p$ *and* $\frac{1}{x^p}$, $k, p, q = 1, 2, 3, \cdots$ | | |
| 22 | $ae^{kx} + by = m$<br>$ce^{kx} + dy = n$ | $x = \dfrac{1}{k}\ln\left(\dfrac{\gamma}{\alpha}\right), \quad \dfrac{\gamma}{\alpha} > 0$<br>$y = \dfrac{\beta}{\alpha}$ |
| 23 | $ae^{kx} + bx^p y = m$<br>$ce^{kx} + dx^p y = n$ | $x = \dfrac{1}{k}\ln\left(\dfrac{\gamma}{\alpha}\right), \quad \dfrac{\gamma}{\alpha} > 0$<br>$y = \dfrac{\beta}{\alpha x^p}$ |
| 24 | $ae^{kx} + b\dfrac{1}{x^p}y = m$<br>$ce^{kx} + d\dfrac{1}{x^p}y = n$ | $x = \dfrac{1}{k}\ln\left(\dfrac{\gamma}{\alpha}\right), \quad \dfrac{\gamma}{\alpha} > 0$<br>$y = \dfrac{\beta x^p}{\alpha}$ |
| 25 | $ae^{kx} + b\sqrt{x}y = m$<br>$ce^{kx} + d\sqrt{x}y = n$ | $x = \dfrac{1}{k}\ln\left(\dfrac{\gamma}{\alpha}\right), \quad \dfrac{\gamma}{\alpha} > 0$<br>$y = \dfrac{\beta}{\alpha\sqrt{x}}$ |
| 26 | $ae^{kx} + b\dfrac{1}{\sqrt{x}}y = m$<br>$ce^{kx} + d\dfrac{1}{\sqrt{x}}y = n$ | $x = \dfrac{1}{k}\ln\left(\dfrac{\gamma}{\alpha}\right), \quad \dfrac{\gamma}{\alpha} > 0$<br>$y = \dfrac{\beta\sqrt{x}}{\alpha}$ |

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| 27 | $a\ln(qx)+by=m$<br>$c\ln(qx)+dy=n$ | $x=\frac{1}{k}e^{\left(\frac{\gamma}{\alpha}\right)}$<br>$y=\frac{\beta}{\alpha}$ |
| 28 | $a\ln(qx)+bx^py=m$<br>$c\ln(qx)+dx^py=n$ | $x=\frac{1}{k}e^{\left(\frac{\gamma}{\alpha}\right)}$<br>$y=\frac{\beta}{\alpha x^p}$ |
| 29 | $a\ln(qx)+\frac{b}{x^p}y=m$<br>$c\ln(qx)+\frac{d}{x^p}y=n$ | $x=\frac{1}{k}e^{\left(\frac{\gamma}{\alpha}\right)}$<br>$y=\frac{\beta x^p}{\alpha}$ |
| 30 | $a\ln(qx)+b\sqrt{x}y=m$<br>$c\ln(qx)+d\sqrt{x}y=n$ | $x=\frac{1}{k}e^{\left(\frac{\gamma}{\alpha}\right)}$<br>$y=\frac{\beta}{\alpha\sqrt{x}}$ |
| 31 | $a\ln(qx)+\frac{b}{\sqrt{x}}y=m$<br>$c\ln(qx)+\frac{d}{\sqrt{x}}y=n$ | $x=\frac{1}{k}e^{\left(\frac{\gamma}{\alpha}\right)}$<br>$y=\frac{\beta\sqrt{x}}{\alpha}$ |
| 32 | $ae^{kx}+b(\ln qx)\,y=m$<br>$ce^{kx}+d(\ln qx)y=n$ | $x=\frac{1}{k}\ln\left(\frac{\gamma}{\alpha}\right)$ , $\frac{\gamma}{\alpha}>0$<br>$y=\frac{\beta}{\alpha\ln(qx)}$ , $qx>0$ |
| 33 | $a\ln qx+be^{kx}\,y=m$<br>$c\ln qx+de^{kx}y=n$ | $x=\frac{1}{p}e^{\left(\frac{\gamma}{\alpha}\right)}$<br>$y=\frac{\beta}{\alpha e^{kx}}$ |
| 34 | $ae^{kx}+bx^p(\ln qx)\,y=m$<br>$ce^{kx}+dx^p(\ln qx)y=n$ | $x=\frac{1}{k}\ln\left(\frac{\gamma}{\alpha}\right)$ , $\frac{\gamma}{\alpha}>0$<br>$y=\frac{\beta}{\alpha x^p\ln(qx)}$ , $qx>0$ |
| 35 | $ae^{kx}+b\frac{1}{x^p}(\ln qx)\,y=m$<br>$ce^{kx}+d\frac{1}{x^p}(\ln qx)y=n$ | $x=\frac{1}{k}\ln\left(\frac{\gamma}{\alpha}\right)$ , $\frac{\gamma}{\alpha}>0$<br>$y=\frac{\beta x^p}{\alpha\ln(qx)}$ , $qx>0$ |
| *Forms involving Trigonometric Functions* , $k,p=1,2,3,\cdots,i=0,1$ | | |
| 36 | $a\sin kx+by=m$<br>$c\sin kx+dy=n$ | $x=i\pi\pm\frac{1}{k}\sin^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y=\frac{\beta}{\alpha}$ |
| 37 | $a\sin kx+bx^py=m$<br>$c\sin kx+dx^py=n$ | $x=i\pi\pm\frac{1}{k}\sin^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y=\frac{\beta}{\alpha x^p}$ |
| 38 | $a\sin kx+b\frac{1}{x^p}y=m$<br>$c\sin kx+d\frac{1}{x^p}y=n$ | $x=i\pi\pm\frac{1}{k}\sin^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y=\frac{\beta x^p}{\alpha}$ |
| 39 | $a\sin kx+b\sqrt{x}y=m$<br>$c\sin kx+d\sqrt{x}y=n$ | $x=i\pi\pm\frac{1}{k}\sin^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y=\frac{\beta}{\alpha\sqrt{x}}$ |
| 40 | $a\sin kx+b\frac{1}{\sqrt{x}}y=m$<br>$c\sin kx+d\frac{1}{\sqrt{x}}y=n$ | $x=i\pi\pm\frac{1}{k}\sin^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y=\frac{\beta\sqrt{x}}{\alpha}$ |
| 41 | $a\cos kx+by=m$<br>$c\cos kx+dy=n$ | $x=i\pi\pm\frac{1}{k}\cos^{-1}\left(\frac{\gamma}{\alpha}\right)$ |

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| | | $y = \frac{\beta}{\alpha}$ |
| 42 | $a\cos kx + bx^p y = m$<br>$c\cos kx + dx^p y = n$ | $x = i\pi \pm \frac{1}{k}\cos^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha x^p}$ |
| 43 | $a\cos kx + b\frac{1}{x^p}y = m$<br>$c\cos kx + d\frac{1}{x^p}y = n$ | $x = i\pi \pm \frac{1}{k}\cos^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta x^p}{\alpha}$ |
| 44 | $a\cos kx + b\sqrt{x}y = m$<br>$c\cos kx + d\sqrt{x}y = n$ | $x = i\pi \pm \frac{1}{k}\cos^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha\sqrt{x}}$ |
| 45 | $a\cos kx + b\frac{1}{\sqrt{x}}y = m$<br>$c\cos kx + d\frac{1}{\sqrt{x}}y = n$ | $x = i\pi \pm \frac{1}{k}\cos^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta\sqrt{x}}{\alpha}$ |
| 46 | $a\tan kx + by = m$<br>$c\tan kx + dy = n$ | $x = i\pi \pm \frac{1}{k}\tan^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha}$ |
| 47 | $a\tan kx + bx^p y = m$<br>$c\tan kx + dx^p y = n$ | $x = i\pi \pm \frac{1}{k}\tan^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha x^p}$ |
| 48 | $a\tan kx + b\frac{1}{x^p}y = m$<br>$c\tan kx + d\frac{1}{x^p}y = n$ | $x = i\pi \pm \frac{1}{k}\tan^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta x^p}{\alpha}$ |
| 49 | $a\tan kx + b\sqrt{x}y = m$<br>$c\tan kx + d\sqrt{x}y = n$ | $x = i\pi \pm \frac{1}{k}\tan^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha\sqrt{x}}$ |
| 50 | $a\tan kx + b\frac{1}{\sqrt{x}}y = m$<br>$c\tan kx + d\frac{1}{\sqrt{x}}y = n$ | $x = i\pi \pm \frac{1}{k}\tan^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta\sqrt{x}}{\alpha}$ |
| *Forms involving reciprocal of Trigonometric Functions* , $k, p = 1, 2, 3, \cdots, i = 0, 1$ | | |
| 51 | $a\csc kx + by = m$<br>$c\csc kx + dy = n$ | $x = i\pi \pm \frac{1}{k}\csc^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha}$ |
| 52 | $a\csc kx + bx^p y = m$<br>$c\csc kx + dx^p y = n$ | $x = i\pi \pm \frac{1}{k}\csc^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha x^p}$ |
| 53 | $a\csc kx + b\frac{1}{x^p}y = m$<br>$c\csc kx + d\frac{1}{x^p}y = n$ | $x = i\pi \pm \frac{1}{k}\csc^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta x^p}{\alpha}$ |
| 54 | $a\csc kx + b\sqrt{x}y = m$<br>$c\csc kx + d\sqrt{x}y = n$ | $x = i\pi \pm \frac{1}{k}\csc^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha\sqrt{x}}$ |
| 55 | $a\csc kx + b\frac{1}{\sqrt{x}}y = m$<br>$c\csc kx + d\frac{1}{\sqrt{x}}y = n$ | $x = i\pi \pm \frac{1}{k}\csc^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta\sqrt{x}}{\alpha}$ |
| 56 | $a\sec kx + by = m$<br>$c\sec kx + dy = n$ | $x = i\pi \pm \frac{1}{k}\sec^{-1}\left(\frac{\gamma}{\alpha}\right)$ |

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| | | $y = \frac{\beta}{\alpha}$ |
| 57 | $a\sec kx + bx^p y = m$<br>$c\sec kx + dx^p y = n$ | $x = i\pi \pm \frac{1}{k}\sec^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha x^p}$ |
| 58 | $a\sec kx + b\frac{1}{x^p}y = m$<br>$c\sec kx + d\frac{1}{x^p}y = n$ | $x = i\pi \pm \frac{1}{k}\sec^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta x^p}{\alpha}$ |
| 59 | $a\sec kx + b\sqrt{x}y = m$<br>$c\sec kx + d\sqrt{x}y = n$ | $x = i\pi \pm \frac{1}{k}\sec^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha\sqrt{x}}$ |
| 60 | $a\sec kx + b\frac{1}{\sqrt{x}}y = m$<br>$c\sec kx + d\frac{1}{\sqrt{x}}y = n$ | $x = i\pi \pm \frac{1}{k}\sec^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta\sqrt{x}}{\alpha}$ |
| 61 | $a\cot kx + by = m$<br>$c\cot kx + dy = n$ | $x = i\pi \pm \frac{1}{k}\cot^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha}$ |
| 62 | $a\cot kx + bx^p y = m$<br>$c\cot kx + dx^p y = n$ | $x = i\pi \pm \frac{1}{k}\cot^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha x^p}$ |
| 63 | $a\cot kx + b\frac{1}{x^p}y = m$<br>$c\cot kx + d\frac{1}{x^p}y = n$ | $x = i\pi \pm \frac{1}{k}\cot^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta x^p}{\alpha}$ |
| 64 | $a\cot kx + b\sqrt{x}y = m$<br>$c\cot kx + d\sqrt{x}y = n$ | $x = i\pi \pm \frac{1}{k}\cot^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta}{\alpha\sqrt{x}}$ |
| 65 | $a\cot kx + b\frac{1}{\sqrt{x}}y = m$<br>$c\cot kx + d\frac{1}{\sqrt{x}}y = n$ | $x = i\pi \pm \frac{1}{k}\cot^{-1}\left(\frac{\gamma}{\alpha}\right)$<br>$y = \frac{\beta\sqrt{x}}{\alpha}$ |

## 3. Conclusion

The proposed Table of Non-linear Simultaneous Equationscompletely eliminates the procedure to solve various forms of such equations. The derived and tabulated solutions could be read-off from the table, similar to the Table of Integrals, and then perform somesimple calculations to obtain the numerical answers. This is especially useful for those who are not able or not willing to solve the non-linear simultaneous equations. Moreover, the elementary arithmetic could be programed into a spreadsheet so, once the constants from the equations are entered, the answers are obtained instantly. In addition, verification of the numerical answers could be done effortlessly using the same spreadsheet.

## References

[1] Larson, Ron, (2010). "Intermediate Algebra". 5th ed., Cengage Learning.

[2] Sullivan, M. (2008). "College Algebra Essentials", 8th ed., Prentice Hall.

[3] Gustafson, RD., Karr, R & Massey, M., (2011). "Intermediate Algebra", 9th ed., Cengage Learning.

[4] C. Vancil, (1996). "College Algebra: A Graphing Approach", Saunders College Publishing.

[5] C. L. Wu and R. J. Adler, (1975)."Nonlinear matrix algebra and engineering applications. Part 1 : Theory and linear form matrix", Journal of Computational and Applied Mathematics, volume I, no 1.

[6] C. G. Broyden, (1968). "A new method of solving nonlinear simultaneous equations", Computing Centre, University of Essex, Wivenhoe Park, Colchester, Essex.

[7] C. Patrascioiu , C. Marinoiu, (2010). "The applications of the non-linear equations systems algorithms for the heat transfer processes", Mathematical methods, computational techniques, intelligent systems.

[8] "Methods for Solving a System of Nonlinear Equations".https://courses.lumenlearning.com/waymakercollegealgebra/chapter/methods-for-solving-a-system-of-nonlinear-equations/

[9] “Simultaneous Equations – One Linear and one Non-Linear”. https://www.youtube.com/watch?v=cWbZqWgsuY8
[10] “How to Solve Nonlinear Systems”. https://www.dummies.com/education/math/calculus/how-to-solve-nonlinear-systems/
[11] “Solving Systems of Nonlinear Equations”. https://math.libretexts.org/Bookshelves/Algebra/Book%3A_Intermediate_Algebra_(OpenStax)/11%3A_Conics/11.06%3A_Solving_Systems_of_Nonlinear_Equations
[12] “Solving Systems of Nonlinear Equations”. https://www.brainfuse.com/jsp/alc/resource.jsp?s=gre&c=36961&cc=108828
[13] “Nonlinear Systems of Equations and Inequalities https://courses.lumenlearning.com/boundless-algebra/chapter/nonlinear-systems-of- equations-and-inequalities/