On Certain Transformations of Poly-Basic Bilateral Hypergeometric Series

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Abstract: ***In this paper, we have established certain transformations of basic hypergeometric series with more than one base. Some of these lead to the relationship between product of two q-series. These results, in turn, lead to very interesting transformations of bi-basic and poly-basic q-series. A few of the results which are representative of the many results obtained are presented in this article.***

Keywords: Bi-basic; Poly-basic; q-series; Hypergeometric series/ function; Transformation; Bilateral hypergeometric series

1. Introduction

A systematic theory of bi-basic hypergeometric series was established by Agarwal and Verma [2,3], yet not much break through could be achieved though a good number of results involving more than one base do exist in the literature.It has been a challenging job to develop a systematic theory of transformations of basic hypergeometric series with several bases. In a series of communications, Denis et al.[4,5], Denis and Singh [6], Singh [9] making use of several series identities and sums of partial series, succeeded in establishing a number of transformations of poly-basic series.

Recently Gasper [7], made use of the following identity:

$$\sum_{k=0}^{n} a_k \sum_{j=0}^{n-k} A_j = \sum_{k=0}^{n} A_k \sum_{j=0}^{n-k} a_j \quad \ldots\ldots\ldots\ldots(1)$$

and using a known indeBnite summation established a transformation of a ${}_{10}\phi_9$with four independent bases. In this paper, we make use of the series identity:

$$\sum_{k=-m}^{n} a_{k+m} \sum_{j=0}^{n-k} A_j = \sum_{k=-m}^{n} A_{k+m} \sum_{j=0}^{n-k} a_j \quad \ldots\ldots\ldots(2)$$

2. Notation and definitions

A basic hypergeometric series is one where each of the parameters is a basic number, with the base being, say, $|q|$¡ 1.A generalization of this series is to have some parameters not all having the same base. By bi-basic hypergeometric series is meant a basic hypergeometric series in which some of the numerator and denominator parameters have the base q and the other numerator/denominator parameters have a diEerent base, say, |q1|¡ 1. A generalized bi-basic hypergeometric function in one variable is defined as

$$\Phi\begin{bmatrix}(a):(b);q,q_1;z\\(c):(d);q^i,q_1^j\end{bmatrix} = \sum_{n=0}^{\infty}\frac{[(a);q]_n[(b);q_1]_n z^n q^{i\,n(n-1)/2} q_1^{j\;n(n-1)/2}}{[q;q]_n[(c);q]_n[(d);q_1]_n}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Where (a) represents the sequence of A-parameters: a1a2···aA, and

$[(a);q]_n = [a_1, a_2, \ldots\ldots a_A,;q]_n = [a_1;q]\ldots\ldots[a_A;q]_n$ with $[a;q]_n = (1-a)(1-aq)(1-aq^2)\ldots\ldots(1-aq^{n-1}), \quad [a;q]_0 = 1.$

The series (3) converges for $(|q|,|q_1|) < 1, |z| < \infty$ when i, j >0 and $(|q|,|q_1|,|z|) < 1$, when i=j=0. We also define a poly-basic hypergeometric series of one variable as

$$\Phi\begin{bmatrix}a_1,a_2,\ldots\ldots a_r: c_{1,1},\ldots,c_{1,r_1};\ldots;c_{m,1},\ldots,c_{m,r_m};q,q_1,\ldots,q_m;z\\ b_1,b_2,\ldots\ldots b_s: d_{1,1},\ldots,d_{1,s_1};\ldots;\ldots;d_{m,1},\ldots,d_{m,s_m}\end{bmatrix}$$
$$= \sum_{n=0}^{\infty}\frac{[a_1,a_2,\ldots\ldots a_r,;q]_n z^n}{[q,b_1,b_2,\ldots\ldots b_s,;q]_n}\prod_{j-1}^{m}\frac{[c_{j,1},\ldots\ldots c_{j,r_m};q_j]_n}{[d_{j,1},\ldots\ldots d_{j,s_j};q_j]_n} \quad \ldots\ldots(4)$$

A sum of terms u_r, where the index r is in the interval $[-\infty;\infty]$ is called a bilateral series, convergent under appropriate conditions, and the series may terminate on either or both sides. Most of the other notations are standard as in [8].

3. Main Transformations

We show how we can establish our main transformations of bilateral basic hypergeometric series through a few selected examples. First we choose to exhibit a simple example: Let us take

$$a_k = \frac{[\alpha,y;q_1]_k q_1^k}{[q_1,\alpha y q_1;q_1]_k} \; and \; A_k = \frac{[\alpha,\beta;q]_k q^k}{[q,\alpha\beta q;q]_k}, \ldots\ldots\ldots(5)$$

In (2) and use the known [1, App.II (8)] partial sum result:

$$2\Phi_1\begin{bmatrix}a,y;q;q\\ayq\end{bmatrix}_N = \frac{[aq,yq;q]_N}{[q,ayq;q]_N}, \quad \ldots\ldots\ldots(6)$$

To get

$$4\Psi_4\begin{bmatrix} \alpha q_1^m, y q_1^m; q^{-n}, q^{-n}/\alpha\beta\,; q_1, q; q_1 \\ q_1^{l+m}, \alpha y q_1^{l+m}; q^{-n}/\alpha, q^{-n}/\beta, \end{bmatrix}$$

$$= \frac{[\alpha,\beta;q]_m[q_1,\alpha y q_1;q_1]_m[\alpha q_1, y q_1;q_1]_n[q,\alpha\beta q;q]_n q^m}{[q,\alpha\beta q;q]_m[\alpha,y;q_1]_m[q_1,\alpha y q_1;q_1]_n[\alpha q,\beta q;q_1]_n q_1^m}$$

$$\times 4\Psi_4\begin{bmatrix} \alpha q^m, \beta q^m; q_1^{-n}, q_1^{-n}/\alpha y\,; q, q_1; q_1 \\ q^{l+m}, \alpha\beta q^{l+m}; q_1^{-n}/\alpha, q_1^{-n}/y, \end{bmatrix} \quad \ldots\ldots\ldots(7)$$

Where the Ψ function represents the bi-basic bilateral series.

To illustrate the power of this method, we give an advanced example: let

$$a_k = \frac{(1-ap^kq^k)(1-bp^kq^{-k})[a,b;p]_k[c,a/bc\,;q]_k q^k}{(1-a)(1-b)[q,aq/b\,;q]_k[ap/c\,,bcp;p]_k}$$

And

$$A_k = \frac{[d, q_1\sqrt{d}, -q_1\sqrt{d}, e, f, g; q_1]_k q_1^k}{[q_1,\sqrt{d},-\sqrt{d}, dq_1/e\,, dq_1/f\,, dq_1/g\,; q_1]_k},\quad (d = efg). \ldots\ldots(8)$$

We get

$$10\Psi_{10}\begin{bmatrix} ap^{m+1}q^{m+1}; bp^{m+1}/q^{m+1}\,; ap^m, bp^m; cq^m, aq^m/bc\,; \\ ap^mq^m; bp^m/q^m\,; ap^{m+1}/c\,; bcp^{m+1}; q^{1+m}, aq^{m+1}/b\,; \\ q_1^{-n}, eq_1^{-n}/d\,, fq_1^{-n}/d\,, gq_1^{-n}/d\,; pq, p/q\,, p, q, q_1; q \\ ; q_1^{-n}/d\,, q_1^{-n}/e\,, q_1^{-n}/f\,, q_1^{-n}\,/\,g \end{bmatrix}$$

$$= \frac{[a;pq]_m[b;p/q]_m[q,cq/b\,;q]_m[ap/c\,,bcp;p]_m}{[apq;pq]_m[bp/q\,;p\,/\,q]_m[a,b;p]_m[c,a/bc\,;q]_m},$$

$$\times \frac{[q_1, dq_1/e\,, dq_1/f\,, dq_1/g\,; q_1]_n[d, q_1\sqrt{d}, -q_1\sqrt{d}, e, f, g; q_1]_m}{[dq_1.\,eq_1, fq_1, gq_1; q_1]_n[q_1,\sqrt{d},-\sqrt{d}, dq_1/e\,, dq_1/f\,, dq_1/g\,; q_1]_m},$$

$$\times \left(\frac{q_1}{q}\right)^m \frac{[ap,bp;p]_n[cq,aq/bc\,;q]_n}{[q,aq/b\,;q]_n[ap/c\,,bcp;p]_n},$$

$$\times 10\Psi_{10}\begin{bmatrix} dq_1^m, q_1^{1+m}\sqrt{d}, -q_1^{1+m}\sqrt{d}, eq_1^m, fq_1^m, gq_1^m; \\ q_1^{1+m}, q_1^m\sqrt{d}, -q_1^m\sqrt{d}, dq_1^{1+m}/e\,, dq_1^{1+m}/f\,, dq_1^{1+m}\,/\,g; \\ ; q^{-n}, bq^{-n}/a\,; cp^{-n}/a\,, p^{-n}/bc; q_1, q, p; q_1 \\ ; q^{-n}/c\,, bcq^{-n}/a\,, p^{-n}/a\,, p^{-n}\,/\,b \end{bmatrix},\quad (d = efg). \quad \ldots\ldots\ldots\ldots\ldots(11)$$

Use these in (1) and make use of the following partial sum [1, App. II (25)]:

$$6\Phi_5\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b\,, aq/c\,, aq\,/\,d \end{bmatrix}_N = \frac{[aq,bq,cq,dq;q]_N}{[q,aq/b\,,aq/c\,,aq\,/\,d;q]_N}, \quad \ldots\ldots(9)$$

And the partial sum [9, App. II (35)]:

$$\sum_{k=0}^{n}\frac{(1-ap^kq^k)(1-bp^kq^{-k})[a,b;p]_k[c,a/bc\,;q]_k q^k}{(1-a)(1-b)[q,aq/b\,;q]_k[ap/c\,,bcp;p]_k} = \frac{[ap,bp;p]_n[cq,aq/bc\,;q]_n}{[q,aq/b\,;q]_n[ap/c\,,bcp;p]_n}, \quad \ldots\ldots\ldots(10)$$

We have established bilateral basic hypergeometric series for the choice of the following:

$$a_k = \frac{(1-ap^kq^k)(1-bp^kq^{-k})[a,b;p]_k[c,a\,/\,bc;q]_k q^k}{(1-a)(1-b)[q,aq\,/\,b;q]_k[ap/c, bcp\,;p]_k}$$

And

$$A_k = \frac{(1-AP^kQ^k)(1-BP^kQ^{-k})[A,B;P]_k[C,A\,/\,BC;Q]_k Q^k}{(1-A)(1-B)[Q,AQ\,/\,B;Q]_k[AP/C,BCP\,;P]_k} \quad \ldots\ldots\ldots\ldots(12)$$

Which in conjunction with the partial sum [10] yields a transformation for a $10\Psi_{10}$.

The following choice:

$$a_k = \frac{(1-ap^kq^k)[a;p]_k[b;q]_k b^{-k}}{(1-a)[q;q]_k[ap\,/\,b;p]_k} \quad and \quad A_k = \frac{(c, q_1\sqrt{c}, -q_1\sqrt{c}, d; q_1)_k}{(q_1,\sqrt{c},-\sqrt{c}, cq_1/d\,; q_1)_k d^k}, \ldots\ldots\ldots\ldots(13)$$

Along with the partial sum result [1, App. II (23)]

$$4\Phi_3\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, e; q; 1\,/\,e \\ \sqrt{a}, -\sqrt{a}, aq/e \end{bmatrix}_N = \frac{[aq,eq;q]_N}{[q,aq/e\,;q]_N e^N}, \quad \ldots\ldots(14)$$

And [9, App. II (34)]

$$\sum_{k=0}^{n}\frac{(1-ap^kq^k)[a;p]_k[c;q]_k c^{-k}}{(1-a)[q;q]_k[ap/c\,;p]_k} = \frac{[ap;p]_n[cq;q]_n c^{-n}}{[q;q]_n[ap/c\,;p]_n} \quad \ldots\ldots\ldots(15)$$

Results in a transformation between a $5\Phi_5$ and a $6\Phi_6$. We have established a number of similar transformation and these will be reported elsewhere and can also be obtained from the authors. The transformation we obatained suggest product theorems. To illustrate, the transformation [7] leads to:

$$2\Psi_2\begin{bmatrix} \alpha q_1^m, y q_1^m; q_1; q_1 x \\ q_1^{1+m}, \alpha y q_1^{1+m} \end{bmatrix} 2\Phi_1\begin{bmatrix} \alpha q, \beta q; q; x \\ \alpha\beta q \end{bmatrix} = \frac{[\alpha,\beta;q]_m[q,\alpha y q_1;q_1]_m}{[q,\alpha\beta;q]_m[\alpha,y;q_1]_m}\left(\frac{q}{q_1}\right)^m$$

$$\times\, 2\Psi_2\left[\begin{matrix}\alpha q^m, \beta q^{1+m}; q; qx\\ q^{1+m}, \alpha\beta q^{1+m}\end{matrix}\right] 2\Phi\left[\begin{matrix}\alpha q_1, \gamma q_1; q_1; x\\ a\gamma q_1\end{matrix}\right], (|q_1 x|, |x|, |qx| < 1). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

It is to be noted that if we take all the bases equal in a particular transformation, then we get the corresponding transformation for a basic hypergeometric function having only one base. Further, specializing the parameters can also lead to interesting results. The final example we present here is to illustrate these aspects.

$$a_k = \frac{[\alpha, \beta; q_1]_k\, q_1^k}{[q_1, \alpha\beta q_1; q_1]_k} \text{ and } A_k = \frac{(1 - ap^k q^k)[a; p]_k [b; q]_k b^{-k}}{(1 - a)[q; q]_k [a\, p/b\, ; p]_k} \quad \ldots\ldots\ldots (17)$$

In (1) and make use of the partial sum results (6) and (15). We get:

$$4\Psi_4\left[\begin{matrix}\alpha q_1^m, \beta q_1^m; q^{-n}; bp^{-n}/a\, ; q_1, q, p; q_1\, /\, b\\ q_1^{1+m}, \alpha\beta q_1^{1+m}; q^{-n}/b\, ; p^{-n}\, /\, a\end{matrix}\right]$$

$$= \frac{[q_1, \alpha\beta q_1; q_1]_m [q; q]_n [ap/b\, ; p]_n [apq; pq]_m [a; p]_m [b; q]_m}{[\alpha, \beta; q_1]_m [ap; p]_n [bq; q]_n [a; pq]_m [q; q]_m [ap/b\, ; p]_m} \times \frac{[\alpha q_1, \beta q_1; q_1]_n b^n}{[q_1, \alpha\beta q_1; q_1]_n (bq_1)^m [q; q]_n [ap/b\, ; p]_n}$$

$$\times\, 5\Psi_5\left[\begin{matrix}ap^{1+m} q^{1+m}, ap^m, bq^m, q_1^{-n}, q_1^{-n}/\alpha\beta\, ; pq, p, q, q_1; 1\, /\, b\\ ap^m q^m; ap^{1+m}/b\, ; q_1^{1+m}; q_1^{-n}/\alpha\, , q_1^{-n}/\beta\end{matrix}\right] \quad \ldots\ldots\ldots\ldots (18)$$

The above transformation (18) suggest the following product theorem:

$$2\Psi_2\left[\begin{matrix}\alpha q_1^m, \beta q_1^m; q_1; q_1 bx\\ q_1^{1+m}, \alpha\beta q_1^{1+m}\end{matrix}\right] 2\Phi\left[\begin{matrix}bq\colon ap; q, p; x\\ -\colon ap\, /\, b\end{matrix}\right]$$

$$= \frac{[q_1, \alpha\beta; q_1]_m [apq; pq]_m [a; p]_m [b; q]_m}{[\alpha, \beta; q_1]_m [a; pq]_m [q; q]_m [ap/b\, ; p]_m (bq_1)^m}$$

$$\times\, 3\Psi_3\left[\begin{matrix}a(pq)^{1+m}\colon ap^m\colon bp^m; pq, p, q; x\\ a(pq)^m\colon ap^{1+m}/b : q^{1+m}\end{matrix}\right] 2\Phi_1\left[\begin{matrix}\alpha q_1, \beta q_1; q_1; bx\\ \alpha\beta q_1\end{matrix}\right] \quad \ldots\ldots\ldots (19)$$

Taking m=0 in (19), we get:

$$2\Phi_1\left[\begin{matrix}\alpha, \beta; q_1; q_1 bx\\ \alpha\beta q_1\end{matrix}\right] 2\Phi_1\left[\begin{matrix}bp\colon ap; q, p; x\\ -\colon ap\, /\, b\end{matrix}\right] = 3\Psi_2\left[\begin{matrix}b\colon apq\colon a; q, pq, p; x\\ -\colon a\colon ap\, /\, b\end{matrix}\right] 2\Phi_1\left[\begin{matrix}\alpha q_1, \beta q_1; q_1; bx\\ \alpha\beta q_1\end{matrix}\right], (|bx|, |x| < 1) \quad \ldots\ldots (20)$$

Furthermore, if we now set $b \to 1$ in (20), we get:

$$2\Phi_1\left[\begin{matrix}\alpha, \beta; q_1; q_1 x\\ \alpha\beta q_1\end{matrix}\right] = (1 - x) 2\Phi_1\left[\begin{matrix}\alpha q_1, \beta q_1; q_1; x\\ \alpha\beta q_1\end{matrix}\right], \quad (|x| < 1). \quad \ldots\ldots (21)$$

To conclude, in this short article we have shown than starting from a modified Gasper [7] identity, it is possible to establish transformation of a poly-basic bilateral hypergeometric series in terms of a similar series, not necessarily having the same number of bases. Only a few examples have been shown here to illustrate our methodology.

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