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On the Development of Quasi-Reflection and Related Properties

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Abstract: ***Reflection of points across lines is commonly taught in schools, but the intuitive qualities of the transformation do not necessarily align with its formal definition. This paper explores the extension of this notion of quasi-reflection, separate from the canonical mathematical sense of the term, by graphical properties. Through the presentation of examples, insight into the relationship of degrees of curves under this process is derived.***

Keywords: differential geometry, reflection, plane curves

1. Introduction and Motivation

The notion of what will be called "quasi-reflection" is considered here. It may help to start with the limiting case, in which quasi-reflection reduces to the conventional reflection, the inspiration for this paper: the reflection of a point across a line. When such a reflection occurs, there are a few important factors, which are the guiding principles of the transformation described by quasi-reflection; a joining line passing through the point, normal to the curve (a line, in this case), and the equidistance of the image and the point to the curve.

There is a significant gap in the mathematical canon of the description of such a process. There is little to no literature following this path towards defining reflection, a shortcoming that this paper hopes to remedy by providing an overview of its basic properties and structure. In exploring this untested and therefore somewhat obscure field, insight may be shed onto the existing relationships between common curves in differential geometry and additional properties related to quasi-reflection.

2. Initial Definitions

There are three main components to the process of quasi-reflection. The first is the *reflectend*, the object that is to be reflected. The second is the *reflector*, which is the differentiable curve or surface across which the reflectend will be reflected. The last is the *image*, which is directly derived from the previous two, and is the resulting object from the quasi-reflection.

Consider first the case where the reflectend is a point (point quasi-reflection).Obtaining the image involves finding all possible normals to the reflector that pass through the point. Having found these, and the points on the curve to which they correspond (the *reflection points*), one projects the reflectend across the reflection points, ensuring that the image and the reflectend are equidistant to the reflection point.

Then, generalizing to a curve as a reflectend, the process is analogous; the point quasi-reflection is applied to each point in the curve, and the resulting locus of points is the image.

In the special case where the reflector is a point, quasi-reflection defined in this manner becomes the same as inversion in a point, since all lines through the reflector are normals.

It should be noted that quasi-reflection is not, in general, an isometry, nor is it even a function. Most accurately, it is a homogeneous binary relation between points in some space. That is, the operation in an arbitrary vector space W can be described as a subset of $W \times W$. This is true given a reflector.

Most generally, then, point quasi-reflection is a binary heterogeneous relation with domain $C^1 \times W$and codomain W, where C^1 is the space of differentiable curves in the appropriate vector space. It is denoted using the Greek qoppa:

$$ϟ \subset (C^1 \times W) \times W \tag{2.1}$$

To summarize, there are two defining principles to point quasi-reflection, given a reflector.

A1. The line joining the image and the reflectend is normal to the reflector.

A2. The distances from the image and from the reflectend to the reflector across this joining line are equal.

The rest of this paper is organized as follows. *Section 3*presents the case of reflecting across a parabola, exploring the nuance that arises from the deviations from canonical notions of reflection. *Section 4* looks at the case of the reflector being the unit circle. *Section 5* details the adjacent idea of the evolute and its property of classifying points based on the number of images they have. *Section 6* explores quasi-reflection in higher-dimensional spaces, briefly looking at the example of the paraboloid. *Section 7*summarizes results on the relationship between degrees of the reflectend and reflector. *Section 8* concludes the paper with a summary of the conclusions and insights obtained in this work and a brief discussion on the future course of this research. Henceforth, all references to "reflection" are to quasi-reflection, unless otherwise specified.

3. Reflecting Across a Parabola

The specific case that will be illustrated here is that where the reflector is the parabola with equation $y = x^2$, with vector space $W = \mathbb{R}^2$. First, point quasi-reflection will be derived. Consider the general point $P = (a_1, a_2)$ and a potential reflection point $R = (x_0, y_0)$. The *reflection point* is the point on the reflector across which reflection is carried out; it is the midpoint of the line segment joining the image and the reflectend. For the parabola, $y_0 = x_0^2$.

Then, condition A1 gives that the following relationship must be true:

$$2x_0 \cdot \frac{x_0^2 - a_2}{x_0 - a_1} = -1 \tag{3.1}$$

And the condition A2 gives that the image can be expressed as follows:

$$(2x_0 - a_1, 2x_0^2 - a_2) \tag{3.2}$$

Manipulating Eq. (3.1) gives:

$$x_0^3 + \left(\frac{1 - 2a_2}{2}\right)x_0 - \frac{a_1}{2} = 0 \tag{3.3}$$

And so, the property earlier stated is discovered; there can be multiple images of a given point under quasi-reflection. In this case, there can be up to 3 distinct images.

It is useful for the intuition to be able to visually place one such process. As such, setting the reflectend to $(3,0)$, Figure 1 is obtained after solving the cubic in Eq. (3.3):

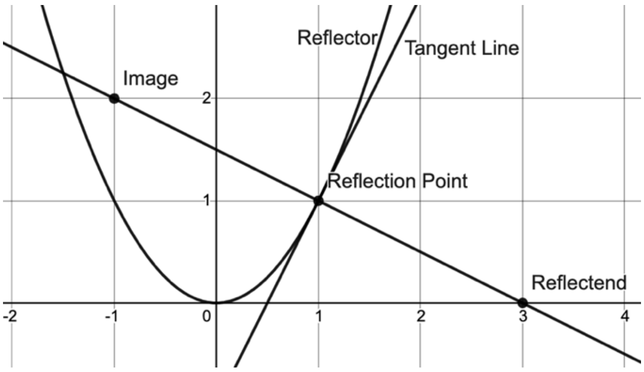

Figure 1

Take for example the point $\left(0, \frac{3}{2}\right)$. There are three reflection points: $(-1,1)$, $(0,0)$, and $(1,1)$. They result in the distinct images $\left(-2, \frac{1}{2}\right)$, $\left(0, -\frac{3}{2}\right)$, and $\left(2, \frac{1}{2}\right)$, as shown below in Figure 2.

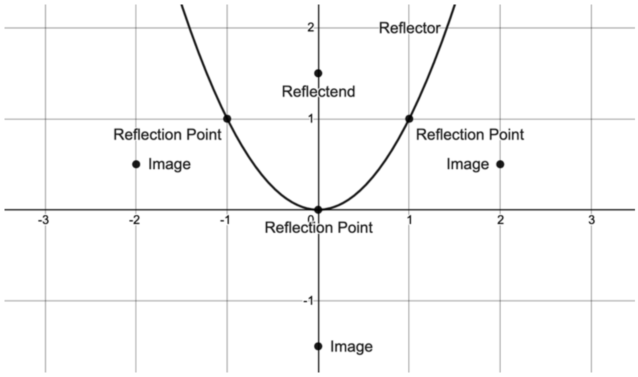

Figure 2

3.1 Reflecting $y = 0.5$

It is now possible to reflect an entire line across this parabola. The case of a horizontal line is illustrated here (this specific case is convenient for the resulting cubic). In this case, A1 gives:

$$x_0^3 - \frac{a_1}{2} = 0 \tag{3.4}$$

The general image for any point $\left(a_1, \frac{1}{2}\right)$ is then:

$$\left(2\sqrt[3]{\frac{a_1}{2}} - a_1, 2\sqrt[3]{\frac{a_1^2}{4}} - \frac{1}{2}\right) \tag{3.5}$$

Finally, then, solving to eliminate a_1 to obtain the equation of the reflected line, the following is obtained:

$$2\left(\frac{2y+1}{4}\right)^{\frac{3}{2}} = a_1 \tag{3.6}$$

So, substituting into the expression for the first co-ordinate,

$$x = 2\left(\frac{2y+1}{4}\right)^{\frac{1}{2}} - 2\left(\frac{2y+1}{4}\right)^{\frac{3}{2}}$$
$$16x^2 = 8y^3 - 20y^2 + 6y + 9 \tag{3.7}$$

This gives us an expression for the equation of the image. Observe that, in reflecting a linear equation across a quadratic equation, a third-degree equation was obtained. Figure 3 illustrates this reflection, with the image represented by the dashed curve.

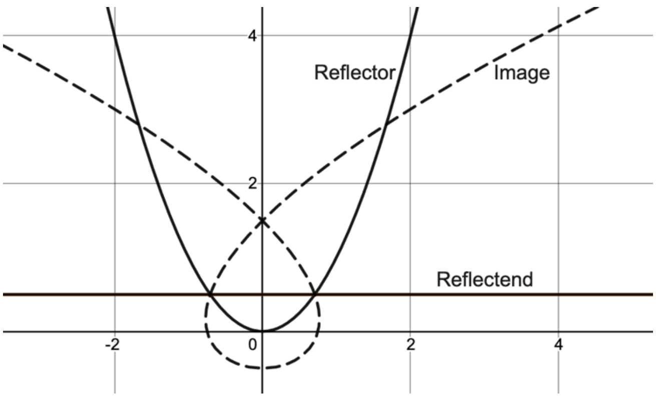

Figure 3

3.2 Reflecting $x = 0$

Another case that of the vertical line, is now illustrated. In particular, the reflectend is the axis of the reflector; this case will be further generalized in *Section 6*. Eq. (3.3) now becomes:

$$x_0^3 + \left(\frac{1-2a_2}{2}\right) x_0 = 0 \tag{3.8}$$

As such, the roots are as follows:

$$x_0 = 0, \pm\sqrt{\frac{2a_2 - 1}{2}} \tag{3.9}$$

The solution of zero corresponds to the reflection across the vertex of the parabola (whose normal coincides with the axis by definition). It is easy to see that this will result in the axis itself in the image. The other images can be expressed as follows:

$$\left(\pm 2\sqrt{\frac{2a_2 - 1}{2}}, a_2 - 1\right) \tag{3.10}$$

The equation of the points described by this parameter then:

$$\frac{x^2+2}{4} = y + 1$$
$$y = \frac{x^2}{4} - \frac{1}{2} \tag{3.11}$$

The image when the reflectend is the axis is then the union of the transformed parabola with the original axis $x = 0$. This is represented in Figure 4.

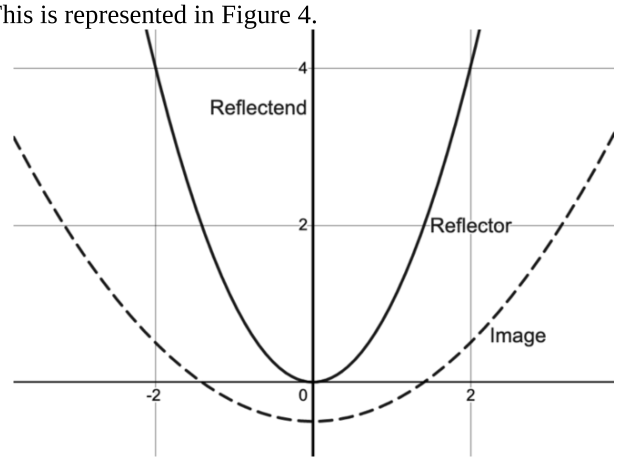

Figure 4

3.3 Reflecting $y = x^2$

The case here is that of the parabola itself; the reflectend is the same as the reflector. While this is typically trivial in the canonical sense of reflection, yielding an image that equals the reflectend, quasi-reflection means that this *auto-reflection* of the parabola is more complicated.

The general point is now (a_1, a_1^2), so the usual cubic in Eq. (3.3) is now as follows.

$$x_0^3 + \left(\frac{1-2a_1^2}{2}\right) x_0 - \frac{a_1}{2} = 0 \tag{3.2}$$

It is clear that the horizontal co-ordinate of the point itself, a_1, is always a root of this equation. Factoring Eq. (3.12) yields the equation below.

$$(x_0 - a_1)\left(x_0^2 + a_1 x_0 + \frac{1}{2}\right) = 0 \tag{3.3}$$

The first root corresponds to the image that is the reflectend itself; this will always be part of the image, but, under quasi-reflection, this is not necessarily the only image. Solving the quadratic obtains the value of the other two roots. The co-ordinates of the reflection points are stated below.

$$x_0 = -\frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 2}}{2} \tag{3.14}$$
$$y_0 = \frac{a_1^2 - 1}{2} \mp \frac{a_1\sqrt{a_1^2 - 2}}{2} \tag{3.15}$$

The expression for the co-ordinates of the image(x', y') is then as below, using Eq. (3.2).

$$x' = -2a_1 \pm \sqrt{a_1^2 - 2} \tag{3.1}$$
$$y' = -1 \mp a_1\sqrt{a_1^2 - 2} \tag{3.2}$$

Eliminating the parameter from these equations through repeated squaring leaves the following result for the equation of the image curve.

$$(x^2 - 4y - 7)^2 = 25(y^2 + 2y + 2) \tag{3.3}$$

Thus, the image is the union of this fourth-degree curve with the original parabola, as shown in Figure 5.

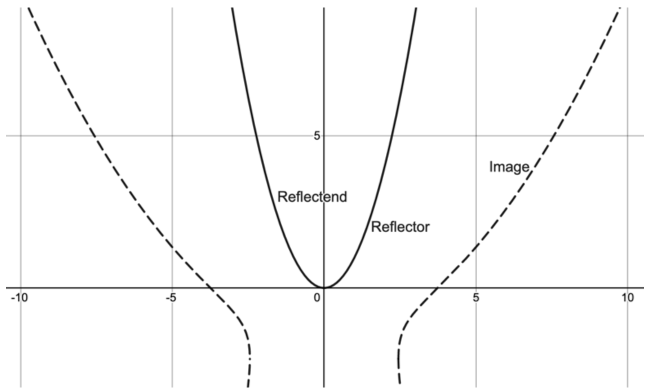

Figure 5

3.4 Classification by number of images

It is natural to turn to the question of classifying all points in the plane by the number of images they have under this operation. Return to the original cubic in Eq. (3.3), restated below:

$$x_0^3 + \left(\frac{1-2a_2}{2}\right) x_0 - \frac{a_1}{2} = 0 \tag{3.3}$$

Consider the cubic discriminant, which is negative when there is one real root, zero when there are two distinct real roots, and positive when there are three distinct real roots. This is as follows, for the general depressed cubic$x^3 + cx + d$:

$$\Delta_3 = -4c^3 - 27d^2 \tag{3.19}$$

Setting equal to zero will give us a boundary, above which points will have three real roots, on which points will have two real roots, and below which they will have one real root. This yields:

$$-4\left(\frac{1-2a_2}{2}\right)^3 - 27\left(-\frac{a_1}{2}\right)^2 = 0 \qquad (3.20)$$

Substituting for the co-ordinates in order to obtain an equation in the Cartesian plane:

$$2(2y-1)^3 - 27x^2 = 0 \qquad (3.4)$$

$$y = 3\sqrt[3]{\frac{x^2}{16}} + \frac{1}{2} \qquad (3.5)$$

This curve will henceforth be referred to as C_1.An important exception to note is that the point $\left(0,\frac{1}{2}\right)$ has only one distinct image, though it does have two real roots. This is due to a repeated root case, but is assured only to exist at singularities such as the cusp of this curve. It is a rotated semicubical parabola shifted by $\frac{1}{2}$ up the vertical axis.It is also the evolute of the reflector, a note that will be pertinent later. Note that the first point reflected in Section 3was below C_1, hence the existence of only one image. Figure 6 illustrates examples of the three possible cases below.

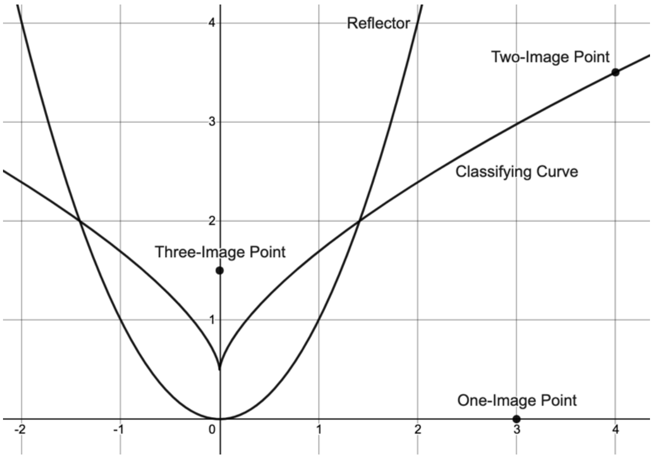

Figure 6

3.5 Reflecting C_1

The reflection of C_1 is relatively simple to derive, since each point reflection results in exactly two images. In general, for a depressed cubic$x^3 + cx + d$ with a repeated root r_1 and other root r_2, the following expression holds.

$$r_1 = \sqrt[3]{\frac{d}{2}}, r_2 = -2\sqrt[3]{\frac{d}{2}} \qquad (3.23)$$

The proof is as follows. Equating equivalent forms of the cubic equation yields:

$$x^3 + cx + d = (x - r_1)^2(x - r_2) \qquad (3.24)$$

Equating the coefficients of the quadratic and constant terms,

$$0 = 2r_1 + r_2$$

$$r_2 = -2r_1 \qquad (3.6)$$

$$d = -r_1^2 r_2 \qquad (3.7)$$

Substituting Eq. (3.25) into Eq. (3.26) yields the following.

$$d = 2r_1^3$$

$$r_1 = \sqrt[3]{\frac{d}{2}} \qquad (3.8)$$

This is as was desired; simple substitution verifies the expression for r_2.

Applying this formula to the cubic in Eq. (3.3), the following values for the reflection point's horizontalco-ordinate are obtained.

$$x_0 = -\sqrt[3]{\frac{a_1}{4}}, \sqrt[3]{2a_1} \qquad (3.28)$$

For the first value, the co-ordinates of the image are as follows, using the equation of C_1.

$$\left(-\sqrt[3]{2a_1} - a_1, -\frac{1}{2}\sqrt[3]{\frac{a_1^2}{2}} - \frac{1}{2}\right) \qquad (3.29)$$

For the second value, the expression is the following.

$$\left(2\sqrt[3]{2a_1} - a_1, \frac{5}{2}\sqrt[3]{\frac{a_1^2}{2}} - \frac{1}{2}\right) \qquad (3.30)$$

Solving for their Cartesian equation shows that the image when the reflectend is C_1 is the union of the following two equations, which correspond to the first and second values above respectively.

$$16y^3 + 8y^2 + x^2 = 0$$

$$16y^3 - 136y^2 + 252y - 125x^2 + 162 = 0 \qquad (3.31)$$

Therefore, the reflection of a third-degree equation, the semicubical parabola, across a quadratic can yield two third-degree equations for the image. This is shown in Figure 7.

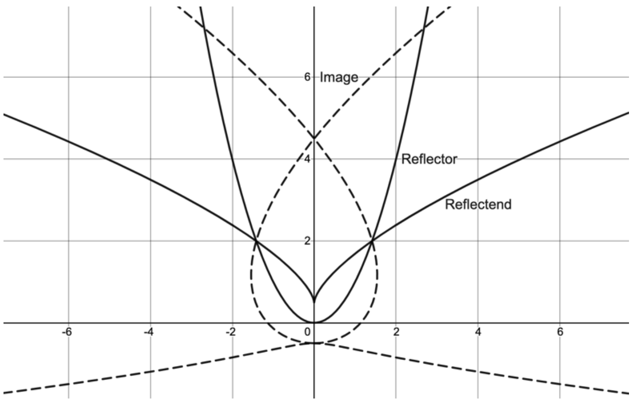

Figure 7

4. Reflecting Across A Circle

Another case study is now shown, one that is simpler in some respects, but more complex in others. Reflection across the unit circle in the Cartesian plane is now considered; generalization to any circle is trivial.

It is obvious that the transformation for any point is simply a multiplication of both co-ordinates by some factor; the desired normal will always be the line through the origin and the reflectend. Every point will have precisely two images corresponding to each "side" of the circle, save for the origin, whose image will be considered later. Geometrically, the equation of the normal for any non-origin point $P = (a_1, a_2)$ is:

$$a_1 y = a_2 x \tag{4.1}$$

Then, the reflection points will be as follows (effectively normalizing the position vector).

$$\left(\frac{a_1}{\sqrt{a_1^2 + a_2^2}}, \frac{a_2}{\sqrt{a_1^2 + a_2^2}}\right)$$
$$\left(\frac{-a_1}{\sqrt{a_1^2 + a_2^2}}, \frac{-a_2}{\sqrt{a_1^2 + a_2^2}}\right) \tag{4.2}$$

These yield the images after applying the condition A2. Expressed more concisely, they are scaled versions of the reflectend by the factors f_1 and f_2, whose values are stated below.

$$f_1 = \frac{2}{\sqrt{a_1^2 + a_2^2}} - 1$$
$$f_2 = \frac{-2}{\sqrt{a_1^2 + a_2^2}} - 1 \tag{4.3}$$

4.1 Reflecting the origin

The case of the origin is now considered. Every line through the origin is a normal. For each, the reflection point is the intersection with the unit circle. The distance from the reflectend to the reflection point is then always 1. Thus, the distance from the reflectend to every possible image must be 2. It then becomes clear that the following equation gives the image of $(0,0)$, which is a circle of radius 2 centered at the origin.

$$x^2 + y^2 = 4 \tag{4.4}$$

4.2 Reflecting a circle

The reflection of a circle centered at the origin across the unit circle is fairly simple; it is clear that the multiplying factors in Eq. (4.3) are constant across the reflectend in this case. They reduce to the following in terms of the radius r of the circle.

$$f_1 = \frac{2}{r} - 1$$
$$f_2 = -\frac{2}{r} - 1 \tag{4.5}$$

As such, the images are two circles of radii with magnitudes rf_1 and rf_2. This simplifies to the following expressions for the two image radii.

$$r_1 = |r - 2|$$
$$r_2 = r + 2 \tag{4.6}$$

4.3 Reflecting a line

A full description of the image of any point has now been given. Naturally, one turns to the images of possible lines. It is immediately clear that the image of any line through the origin is simply the reflectend itself.

Further manipulation in order to find general equations of the images of other lines directly is algebraically difficult. To simplify matters, radial symmetry can be leveraged; only lines in a certain orientation must be considered, which can then be rotated. For ease, the lines of the form $x = c$ are considered. In this case, the following equations are valid for the image curve.

$$y = a_2 \frac{x}{c} \tag{4.7}$$
$$x = c\left(\frac{\pm 2}{\sqrt{c^2 + a_2^2}} - 1\right) \tag{4.8}$$
$$(x + c)^2 = \frac{4c^2}{c^2 + a_2^2}$$
$$a_2^2 = \frac{4c^2}{(x + c)^2} - c^2 \tag{4.9}$$

Eliminating the parameter, the following explicit equation can be derived for the image by substituting Eq. (4.9) into Eq. (4.7) after squaring.

$$y^2 = \frac{x^2}{c^2}\left(\frac{4c^2}{(y + c)^2} - c^2\right)$$
$$(x^2 + y^2)(x + c)^2 = 4x^2 \tag{4.10}$$

Then, for the arbitrary line described by its minimum distance r from the origin and the angle θ it makes with the positive horizontal axis, an anticlockwise rotation of Eq. (4.10) by $\theta + \frac{\pi}{2}$ is required. Using the rotation matrix, the following expression is valid for this transformation.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.11}$$

$$x' = -x\sin\theta + y\cos\theta \tag{4.12}$$

Observe that the first bracket in the left-hand side of Eq. (4.10) is independent of rotation, representing the square of the distance to the origin. Therefore, the general image for a line parameterized in this manner is as follows.

$$x^2 + y^2 = 4\left(\frac{-x\sin\theta + y\cos\theta}{-x\sin\theta + y\cos\theta + r}\right)^2 \tag{4.13}$$

Given this expression, it is easy to convert it for the slope-intercept form. The following equations aid in this conversion, for the equation $y = mx + c$. They are derived from basic co-ordinate geometry.

$$r = \frac{c}{\sqrt{1 + m^2}} \tag{4.14}$$
$$\theta = \tan^{-1} m \tag{4.15}$$

This yields, after some manipulation, the following result, having substituted in Eq. (4.13).

$$x^2 + y^2 = 4\left(\frac{-mx + y}{-mx + y + c}\right)^2 \tag{4.16}$$

4.4 Asymptotic behavior

A separate conclusion that can be drawn is that it approximates asymptotically to inversion about the origin. That is to say, the image of the line with equation $ax + by = c$ has the asymptote as follows.

$$ax + by = -c \tag{4.17}$$

The proof is simple; using the expressions for the multiplying factors in Eq. (4.3), it can be seen that the following is true as the reflectend grows far from the origin.

$$f_1 \to -1$$
$$f_2 \to -1 \quad (4.18)$$

Thus, it must be the case that both images tend towards an inversion through the origin. One possible case of the reflection of the line with equation $y = x - 1$ is illustrated in Figure 8; the asymptotic behavior can be observed. The equation of the image is as follows.

$$x^2 + y^2 = 4\left(\frac{y - x}{y - x - 1}\right)^2 \quad (4.19)$$

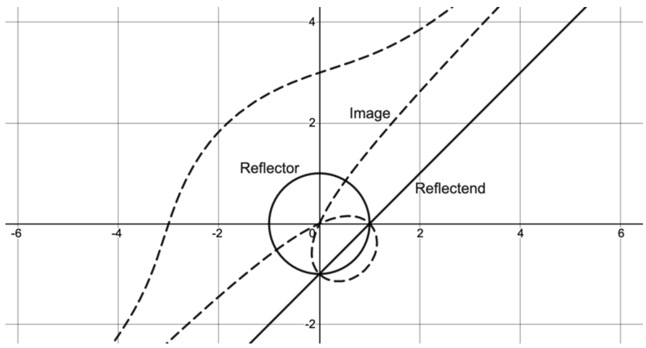

Figure 8

5. The evolute and classification by Number of Images

More general results that will be valid for any possible reflector are now presented. More specifically, the classification of points based on the number of images they produce will be considered.

It was observed in Section 3.3 that it was the evolute which determined the number of images a given point had under the reflection across the parabola. Since all parabolas are affine transformations of that considered earlier, it is easy to see that this must be the case for parabolas in general.

It is required that a formal process is described, through which one may obtain the number of images of a given point for the arbitrary reflector in order to progress further.

Given an arbitrary parameterized curve $\gamma(t) = \big(x(t), y(t)\big)$ in the two-dimensional plane, the normals are characterized by the following equivalent equations.

$$\big(\vec{x} - \gamma(t)\big) \cdot \gamma'(t) = 0 \quad (5.1)$$

$$y'(t)\big(y - y(t)\big) = -x'(t)\big(x - x(t)\big) \quad (5.2)$$

Therefore, the condition for some point $\gamma(t_0)$ to be a reflection point given some reflectend (a_1, a_2) is obtained by substituting into Eq. (5.2) (5.1).

$$y'(t_0)\big(a_2 - y(t_0)\big) = -x'(t_0)\big(a_1 - x(t_0)\big) \quad (5.3)$$

One may then consider the general polynomial with equation $y = Q(x)$. For this class of reflector curves, it is the case that the classifying curve is simply the evolute. This can be proven by considering the evolute as the envelope of normals, an alternative definition to the union of centers of curvature.

To find the evolute in this way, it is required that an expression is found for the general normal through the point $\big(t, Q(t)\big)$. This is as follows.

$$-Q'(t)\big(y - Q(t)\big) = (x - t) \quad (5.4)$$

Now, define $F(x, y, t)$ so that equating to zero gives Eq. (5.4).

$$F(x, y, t) = -Q'(t)\big(y - Q(t)\big) + x - t \quad (5.5)$$

Solving for the envelope requires the following condition to be true [1].

$$\frac{\partial F}{\partial t} = F = 0 \quad (5.6)$$

This effectively means a double root, so an equivalent step would be to set the discriminant of the function in t to zero. But this is the same as the condition required for the classifying equation, since it is derived by setting the discriminant to zero when $t = x_0$ (which holds in this case). Hence, the evolute is the classifying curve when the reflector is a polynomial curve.

Observe that this method does not extend to the general curve, since the discriminant connection does not exist. While a double root is still required for the evolute, it is not certain that points above and below the curve will behave in the same fashion as with the polynomial discriminant.

An interesting consideration that arises from this work is the maximum number of images a point may have given a reflector. For a polynomial with degree n, it is clear that this is $2n - 1$ by considering the degree of the right-hand side of Eq. (5.5).

Another supporting fact of the connection between degree and the maximum number of images can be found when the reflector is set to be a sinusoidal curve. In this case, the equation $y = \cos x$ will be considered for simplicity. It is true that there is no maximum number of images for this case. The proof is as follows.

Let the reflectend be the point $(0, a_2), a_2 > 0$. Let $m_l(x_0)$ be the value of the slope of the line joining the origin and a candidate reflection point $(x_0, \cos x_0)$, and let $m_c(x_0)$ be the value of the normal slope to this candidate point. Consider the domain such that $x_0 \in \left]2\pi n + \pi, 2\pi n + \frac{3\pi}{2}\right], n > 0$. Then, m_l can be tightly bounded by the following inequality. It is also a strictly increasing function in the horizontal co-ordinate.

$$\frac{-a_2 - 1}{2\pi n + \pi} < m_l \leq -\frac{a_2}{2\pi n + \frac{3\pi}{2}} \quad (5.7)$$

On the other hand, m_c is tightly bounded by the following inequality and is unbounded from below. It is a strictly increasing function in the horizontal co-ordinate.

$$m_c \leq -1 \quad (5.8)$$

Consider, then, their difference $d(x) = m_c(x) - m_l(x)$ over the same range. This is also unbounded from below and continuous, but takes on the final value as below.

$$d\left(2\pi n + \frac{3\pi}{2}\right) = \frac{a_2}{2\pi n + \frac{3\pi}{2}} - 1 \qquad (5.9)$$

Then, if this value is greater than 0, the intermediate value theorem guarantees that there will be at least one point in the range such that $m_l(x_0) = m_c(x_0)$. This will be a valid reflection point, and therefore will generate a valid image.As such, it is clear by construction that the choice of a high enough value of a_2 will yield as many images as desired.

Why does this align with the previous intuition between the degree and maximum number of images? If one considers the Taylor expansion of the cosine function, it may be said that the degree is, in an informal sense, infinite. As such, the infinitude of the maximum number of images is also appropriate.

6. Higher Dimensional Reflection

The most general description of quasi-reflection outlined in Section 2 extends very well to higher dimensions. The derivation of explicit equations can grow increasingly laborious, however, requiring the elimination of more parameters, sometimes analytically impossible, as dimensions grow larger than 2.

In the case of three dimensions, visualization remains helpful; the normal is simply the line through a given candidate reflection point which is normal to the tangent plane of a given three-dimensional surface. While visual intuition breaks down beyond this, the notion of the hyperplane allows us to describe the general normal. It is clear the reflector must be of dimension one less than the space in which it exists for quasi-reflection to remain well-defined.

An illustrative finding will be given here in higher dimensions. This is the dimensionally general case of the reflection in which the reflector is a hyperparaboloid and the reflectend is the axis of this hyperparaboloid.

The equation of the n-dimensional paraboloid to be considered is as follows, relabeling the variables for ease.

$$y = \sum_{i=1}^{n} x_i^2 \qquad (6.1)$$

Of course, the axis itself will be part of the image, resulting from the universal reflection point at the origin. To find the other reflection points, using the linear order Taylor approximation near a given point $R = (b_1, b_2, \cdots, b_n, y_0)$ to represent the tangent hyperplane, the equation of this is:

$$y = \sum_{i=1}^{n} 2b_i(x_i - b_i) + y_0 \qquad (6.2)$$

Since R is on the n-dimensional paraboloid, the following relationship holds.

$$y_0 = \sum_{i=1}^{n} b_i^2 \qquad (6.3)$$

The vector normal to this hyperplane is then represented by the following equation, extracting the coefficients from the Cartesian expression in Eq. (6.2).

$$(2b_1, 2b_2, \cdots, 2b_n, -1)^T \qquad (6.4)$$

It is needed that the point P on the axis with "vertical" co-ordinate a_n and other co-ordinates equal to zero is such that the difference between R and P is parallel to this normal. This simplifies to the following, using Eq. (6.3).

$$a_n = y_0 + \frac{1}{2} = \sum_{i=1}^{n} b_i^2 + \frac{1}{2} \qquad (6.5)$$

Geometrically, the locus of reflection points is a hypersphere of $n-1$ dimensions as all points with the given vertical co-ordinate on the n-dimensional paraboloid will be valid reflection points. Similarly, the image is a similar sphere with double the radius. The equation of the image of the general axis point is then as follows.

$$\sum_{i=1}^{n} x_i^2 = 4a_n - 2, y = a_n - 1 \qquad (6.6)$$

Thus, solving to eliminate the parameter yields the image of the axis as the union of the axis and a transformed n-dimensional paraboloid, whose equation is given below.

$$y = \frac{1}{4}\sum_{i=1}^{n} x_i^2 - \frac{1}{2} \qquad (6.7)$$

This aligns with the case in two dimensions that was verified in Section 3.

7. Degrees of Components

A relationship between the degrees of the equations of the reflectend, the reflector, and the image, if it exists would be a valuable finding. Whilst it is beyond the scope of this paper to derive such a relationship, possible combinations that have been derived either earlier, or exist previously in the mathematical canon are presented below. All further findings in this section are in $\mathbb{R}^2$ only.

It first must be considered precisely what is meant by degree in this context. Restricting quasi-reflection to the two-dimensional case, the degree of a curve refers to the degree of the expression $g(x, y)$, where the equation of the curve can be written as $g(x, y) = 0$.

An important case is, for example, the image of the axis of the parabola in Section 3. While it was described as the union of a degree-one and degree-two equation, this can be encapsulated by the degree-three equation as follows:

$$x\left(y + \frac{1}{2} - \frac{x^2}{4}\right) = 0 \qquad (7.1)$$

The union can thus be obtained simply by multiplying, so that there is a unique degree corresponding to each image which is the sum of individual degrees. Though operations such as squaring both sides arbitrarily increase the degree while keeping the equation constant, the fully "reduced" form will be considered here; that which has the minimum possible degree while maintaining integer exponents.

Denote a quasi-reflection with fixed reflector γ by a subscript of this reflector: $\lightning_{\gamma}$. Now denote the induced relation between degrees by $R_{\gamma} \subseteq \mathbb{N}_0 \times \mathbb{N}_0$.

There is a notable property that should be mentioned, shown as follows. Holding the reflector constant, the following is true

$$p_1 R_{\gamma} q_1 \land p_2 R_{\gamma} q_2 \Rightarrow (p_1 + p_2) R_{\gamma} (q_1 + q_2) \tag{7.2}$$

Taking the unions of the reflectends and the images provided neither is identical proves this. Thus, only quasi-reflections that cannot be derived in this trivial manner are presented in the table below.

Reflected Degree	Reflector Degree	Image Degree
n	0	n
n	1	n
1	2	1
1	2	3
1	2	4
2	2	4
2	2	6
3	2	6

The first two rows are elementary; the third row was derived in the case of lines through the origin across a circle. The fourth row refers to the reflection of both the axis and the earlier line across the parabola. The fifth row refers to the reflection of lines not passing through the origin across a circle. The sixth row is the reflection of a circle across the unit circle. The seventh row was the reflection of the parabola across itself. The eighth row corresponds to the reflection of C_1 across the parabola.

8. Conclusion and Future Work

It has been shown that the concept of quasi-reflection holds depth. A general description of the process has been given, and specific results for the cases of parabola and circle reflectors have been presented. That the evolute of a curve with equation of one co-ordinate equal to a polynomial function of the other is the classifying curve has also been proven. Reflection of the axis of a hyperparaboloid across the hyperparaboloid itself has also been derived and demonstrated. Finally, insight into the relationship of the degrees of the components under quasi-reflection was presented based on the previous work throughout the examples.

It is hoped that this paper will serve as the foundation for the study of this curious and surprisingly rich field of quasi-reflection. Though quasi-reflection is more esoteric than canonical reflection and occasionally difficult to work with as a consequence of having lost traditionally required criteria, it holds novel challenges and connections as a result. The merit of such work lies not only in the straightforward applications to optics and visual perception in parabolic mirrors, for example, but also in the potential to discover new mathematics.
Future directions for research are numerous and deep, as is natural in a nascent area of study. The relationship between the turning points and the number of images warrants further investigation. The component degree pattern is also suggestive of an underlying structure that is mathematically useful. Beyond these directions touched on within this paper, there are further extensions that exist; translating this process algebraically to the complex field, or defining quasi-reflection rigorously for reflectors of an arbitrary dimension.

References

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