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On Left Invariant (γ, β)- Metric on a Lie Group

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Abstract: ***The present paper is a study of the left-invariant (γ,β)-metrics on Lie groups. There are derived many geometrical objects and theorem based left-invariant metrics.***

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1. Introduction

Let G be a Lie group. There is a relation,

$$a(y, X(X)) = \beta(x, y), \text{ for every } x \in M, y \in T_x M \quad (1.1)$$

The equation (1.1) is as followed that there exists a unique vector field X on M and 1- form β metric on a Riemannian manifold (M, a) [2].

Let a is left invariant Reimannian metric and X is a left-invariant vector field on Lie group G, such that $\|X\|\alpha = a(X, X) \leq b_0$, then the ($\gamma$, β)-metric is left-invariant. The construction of invariant (α, β)-metric verify on the article [3].

The geodesic equation in left-invariant metric given as: [4],

$$\dot{\xi} = ad_{\xi}^{T_A} (\xi) \quad (1.2)$$

where $ad_{\xi} = [\xi,.]$ and $ad_{\xi}^{T_A}$ is the transpose of map ad_{ξ}, with inner product $\langle .,. \rangle_A$. The Koszul formula for left-invariant metric is given by,

$$2\langle \nabla XY, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle \quad (1.3)$$

The sectional curvature defined as

$$K(u, v) = \langle R(u, v) u, v\rangle \quad (1.4)$$

Where

$R(u, v) = \nabla[_{u, v]} - \nabla u \nabla v + \nabla v \nabla u$

2. Preliminaries

The left-invariant (α, β)-metrics on Lie groups discussed by M. Hosseini and H. R. Salimi Moghaddam [5]. Reza Chavosh Khatamy and Uuldoz Ghalebsaz Jedy constructed left-invariant (α,β)-metrics [3].

Consider $M = \frac{G}{H}$ is a reductive homogeneous space [6]. The Riemannian α metric induced on inner product in the cotangent space $T_x^* M$, as $\langle dx^i, dx^j\rangle = a^{ij}(x)$ gives a smooth vector field $\tilde{X}$ on M. The vector field defined on M characterizes as [3]

$$\tilde{X} = (b)^i \frac{\partial}{\partial x_i} \quad (2.1)$$

Where $(b)^i = \sum_{j=1}^{n} a^{ij} b_j = b_i$

$$\langle y, \tilde{X}\rangle = \beta(x, y), \quad (2.2)$$

for $\forall y \in T_x M$. There always exist invariant vector field $\tilde{X}$ on $M = \frac{G}{H}$ [7] Which can be described as

$$\frac{G}{H} \cong V \quad (2.3)$$

where $V = \{X \in M | Ad(h) X = X, \forall h \in H\}$.

Definition 2.1: A Finsler metric $L(x, y)$ is called an (γ, β)-metric, when L is positive homogeneous function $L(\gamma\ \beta)$ of first degree in two variable γ, and β, where $\gamma^3 = a_{ijk}(x)\, y^i y^j y^k$ is cubic metric and $\beta = b_i(x) y^i$ one form metric[8].

There are example of (γ,β)-metrics.

1) $\bar{L}(\gamma^3 + \beta)$
2) $\bar{L}(\gamma + \beta) = \gamma^3 + a\,\gamma + c\beta^2$ where $a, b, c \in R$
3) $\bar{L}(\gamma, \beta) = (\gamma, \beta)^2$

These are regular Lagrangian with (γ, β)- metrics but only (3) is reducible in finsler space other examples represent non finsler space with (γ, β)- metrics.[9]

For a non zero vector $y \in T_x M$ the Berwald curvature $B_y = B^i_{jkl} dx^i \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ and the Berwald tensor define as $B^i_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(x, y)$.

The mean Berwald curvature $E_y = E_{ij}\, dx^i \otimes dx^j$ is defined as $E_{ij} = B^m_{imj}$. When the mean Berwald curvature vanish as: $E_{ij} = 0$, then this Finsler metric weak Berwald metric. There exist isotropic maen Berwal curvature for Finsler metric . F if $E_{ij} = \frac{n+2}{2} cF_{y^i y^j}$ Here $c = c(x)$ is a scalar function on manifold M.

Definition 2.2. Spray: Let M be a manifold. A spray on M is a smooth vector field G on $TM \setminus 0$ expressed in a standerded local coordinate system (x^i, y^i) in TM as follows:

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$$

The Finsler metric F on an open suset $u \in R^n$ is dually flat if and only if it satisfies

$$(F^2)_{x^k y^l y^k} = 2(F^2)_{x^l}$$

Finsler metric $F = F(x, y)$ on a manifold M is said to be locally flat at any point, if there is a coordinate system (x^i) in which the spray coefficient are in the following form,

$$G^i = \frac{1}{2} g^{ij} H_{yj}$$

A Finsler metric weakly Berwald metric if $E_{ij} = 0$.

3. Existence of left-invariant (γ, β)-metric

Reza Chavosh Khatamy and Uuldoz Ghalebsaz Jedy describe the existence of invariant (α, β)-metrics[3]. Consider $M = \frac{G}{H}$ is a reductive homogeneous space.

Here $\mathfrak{g}$ = *Lie G* and $\mathfrak{h}$ = *Lie H*.

Define the inner product as

$$\langle .,. \rangle = \langle [dx^i, dx^j], dx^k \rangle \quad (3.1)$$

where $[dx^i, dx^j] = dx^i dx^j - dx^j dx^i$,

$$\langle [dx^i, dx^j], dx^k \rangle = a^{ijk}. \quad (3.2)$$

Assume that $\mathfrak{m}$ is an orthogonal complement of $\mathfrak{h}$ on $\mathfrak{g}$ in the inner product defined by equation (3.2). Let $\tilde{X} = b^i \frac{\partial}{\partial x^i}$, such that

$$\langle y, \tilde{X} \rangle = b_i(x)\, y^i, \quad (3.3)$$

which is β-metrics (1-form metrics). The $b_i(x)$ calculated by

$$b^i = \sum_{jk}^{nn} a^{ijk} a_{jk}, \quad (3.4)$$

Where $a^{jk} = \langle dx^j, dx^k \rangle$. Hence $\|\beta\|_\alpha = \|X\|$.

Lemma 3.1. There exists a bijection between the set of invariant vector fields on $\frac{G}{H}$ and the subspace V = X ∈ m|$Ad(h)X = X, \forall h \in H$

The proof can be found in [7].

Theorem 3.2. If $\mathfrak{g}$ *is Lie G. Consider invariant (α,β)-metric on* $\mathfrak{g}$. *Then there exist a* (γ, β)*-metric on* $\mathfrak{g}$.

Proof: The $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. So $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Now from equation (3.3) and (3.4) gives invariant vector field on *M*. This invariant vector the field is a bijection on *M* verify by the Lemma (3.1).

Lemma (3.1) gives that there exists an invariant vector the field on $\mathfrak{g}$ by equation

$$\langle Ad(h)X, Ad(h)Y \rangle = \langle X, Y \rangle, \forall X, Y \in \mathfrak{g}, h \in H \quad (3.5)$$

4. Locally dually flat conformally transformed in Lagrange space with (γ,β)-metric

In the study of information geometry on Riemannian manifolds, Amari-Nagaoka developed the notion of dually flat Riemannian metrics [10]. Locally dually flatness for Finsler metrics notion developed by Shen [11]

A transform Finsler metric $\bar{F}$ on a manifold M^n is said to be locally dually flat if $[\bar{F}^2]_{x^k y^l} y^k = 2[\bar{F}]_{x^l}$ at any point with the coordinate system (x^i, y^i) in *TM*
TM.

Let the conformal transformation $\bar{L} = e^\alpha L$, where L is Lagrange metric with (γ, β)- metric. Since $\bar{L}^2_{x} y = e^{2\alpha} [L^2_{x} k + 2F^2 \alpha_k]$, where $\alpha_k = \frac{\partial \alpha}{\partial x^k}$, we have $\bar{L}^2_{x^k y^l} y^k = e^{2\alpha} [L^2_{x^k y^l} y^k + 2L l_l \alpha_k y^k]$

Hence

$$\begin{aligned} L^2_{x^k y^l} y^k &= 2L_\gamma^2 \frac{a_{ij}(y^i y^j y^k)^2}{3\gamma^2} A_k + 2L_\gamma L_\beta b_l A_k y^i y^j (y^k)^2 + 2LL_{\gamma\gamma} \\ &\times \frac{a_{ij}(y^i y^j y^k)^2}{3\gamma^2} A_k + 2LL_{\gamma\beta} b_l A_k y^i y^j (y^k)^2 - \frac{4}{3\gamma^3} A_k (y^i y^j y^k)^2 y^k \\ &+ 2L_\gamma L_\beta \frac{a_{ij}(y^i y^j y^k)^2}{3\gamma^2} + 2L_\beta^2 b_l B_k (y^k)^2 + 2LL_\beta \frac{a_{ij} y^i y^j (y^k)^2}{3\gamma^2} B_k \\ &+ 2LL_{\beta\beta} b_l B_k (y^k)^2 + 2LL_\beta B_l y^k. \end{aligned} \quad (4.1)$$

And

$$2L^2_{x^l} = 4\, LL_\gamma y^i y^j y^k A_l + 4LL_\beta B_{l y^l} \quad (4.2)$$

and

$$2LL_{y^l} \alpha_k y^k = \frac{2}{3\gamma^2} LL_\gamma a_{ij} y^i y^j y^k + 2LL_\beta b_{l y^k})\, \alpha_k \quad (4.3)$$

We have

$$2\, \bar{L}^2_{x^l} - \bar{L}^2_{x^k y^l} y^k = e^{2\alpha}(2L^2_{x^l} + 4L^2 \alpha_l - L^2_{x^k y^l} y^k - 2Ll\beta blyk\, \alpha k \quad (4.4)$$

If Lagrange space is locally dually flat then $2\, L^2_{x^l} - L^2_{x^k y^l} y^k = 0$.Using equation (4.1),(4.2) and (4.3) in equation (4.4) we find out that,

$2\bar{L}^2{}_{x^l} - \bar{L}^2{}_{x^k y^l} y^k = 4L^2\alpha_l - \left(\frac{2}{3\gamma^2} LL_\gamma a_{ij} y^i y^j + 2\, LL_\beta b_l\right) \alpha_k y^k$ (4.5)

Where terms used in equation (4.1) to (4.4) are defined in the following way,

1) $\frac{\partial\gamma}{\partial x^k} = A_k y^i y^j y^k$

2) $\frac{\partial\beta}{\partial x^k} = B_k y^k$

3) $\frac{\partial^2\beta}{\partial y^l \partial x^l} = B_l$

4) $\frac{\partial\gamma}{\partial y^l} = \frac{1}{3\gamma^2} a_{ij} y^i y^j$

Theorem 4.1. *Let $\bar{L}$ be a conformal transformed metric with (γ, β)- metric on a manifolds M^n in Lagrange space. Then, $\bar{L}$ is locally dually flat metric if and only if* $2L\alpha_l L_\gamma a_{ij} y^i y^j + L_\beta b_l)\, \alpha_k y^k = 0$

Corollary 4.1. *If L is locally dually flat metric then the conformally transformed Lagrangian metric $\bar{L}$ is also locally dually flat if and only if conformal transformation homothetic..*

Proof. Form equation (4.5) the L is locally dually flat if and only $2L\,\alpha_l - \left(\frac{1}{3\gamma^2} L_\gamma a_{ij} y^i y^j + L_\beta b_l\right)\alpha_k y^k = 0$. Hence $\bar{L}$ is locally dually flat if and only if

$$\alpha_l L - \left(\frac{1}{3\gamma^2} L_\gamma a_{ij} y^i y^j + L_\beta b_l\right) = \alpha_0 \quad (4.6)$$

Contracting equation (4.6) with y^l, We have

$$\alpha_0 L - \left(\frac{1}{3}\gamma L_\gamma + L_\beta \beta\right)\alpha_0 = 0$$

This gives $\alpha_0 = 0$. Hence from equation (4.6) , $\alpha_l = 0$, i.e. $\frac{\partial\alpha}{\partial x^l} = 0$. So α is constant. Therefore the transformation is homothetc. The converse is also true.

5. Conformally transformed Lagrangian (γ, β)-metric with isotropic E-curvature

The Berwald curvature of $\bar{L}$ is defined as

$$\bar{B}^i_{jkl} = \frac{\partial^3 \bar{G}^i}{\partial y^j \partial y^k \partial y^l} \quad (5.1)$$

Where $\bar{G}^i$ are spray coefficients of a Lagrange space $\bar{L}$. The trace of the Berwald curvature is called the E-curvature. So $\bar{E}_{ij} = \frac{1}{2}\bar{B}^m_{mij}$. Let $\bar{L}$ is a Lagrangian metric on an n-dimensional manifold M^n. Then the isotropic mean Berwald curvature or of isotropic E-curvature defined as

$$\bar{E}_{ij} = \frac{c\,(n+1)}{2\bar{L}}\, \bar{h}_{ij} \quad (5.2)$$

.

where $\bar{h}_{ij} = \bar{g}_{ij} - \bar{l}_i \bar{l}_j$ is the angular metric and $c = c(x)$ is a scalar function on M^n. Now $\bar{L}$ will be weakly Berwald metric if scalar function $c = 0$. In view of equation (3.4) the angular metric is given by

$$\bar{h}_{ij} = e^{2\alpha}\left\{\rho\alpha_{ij} + \rho - 2a_i a_j + \rho - 1\left(a_i b_j + a_j b_i\right) + \rho_0 b_i b_j - \frac{L_\gamma^2}{9\gamma^4} a_i a_j - \frac{L_\gamma L_\beta}{3\gamma^2}\left(a_i b_j + a_j b_i\right) - L_\beta^2 b_i b_j\right\} \quad (5.3)$$

From equation (5.2) and (5.3) , we have

$$\bar{E}_{ij} = \frac{(n+1)c}{2\bar{L}} e^{\alpha}\left\{\rho\alpha_{ij} + \rho_{-2}\, a_i a_j + \rho_{-1} - \left(a_i b_j + a_j b_i\right) + \rho_0 b_i b_j - \frac{L_\gamma^2}{9\gamma^4} a_i a_j - \frac{L_\gamma L_\beta}{3\gamma^2}\left(a_i b_j + a_j b_i\right) - L_\beta^2 b_i b_j\right\} \quad (5.4)$$

After simplification the equation (5.4) we have

$$\bar{E}_{ij} = \frac{(n+1)c}{2L} e^{\alpha}\left\{\rho\alpha_{ij} + \left(\rho_{-2} - \frac{L_\gamma^2}{9\gamma^4}\right) a_i a_j + \left(\rho - 1 - \frac{L_\gamma L_\beta}{3\gamma^2}\right)\left(a_i b_j + a_j b_i\right) + \left(\rho_0 - L_\beta^2\right) b_i b_j\right\} \quad (5.5)$$

The equation (5.5) shows that $c = 0$, because neither $e^{\alpha} = 0$ nor $\rho a_{ij} + \left(\rho_{-2} - \frac{L_\gamma^2}{9\gamma^4}\right) a_i a_j + \left(\rho - 1 - \frac{L_\gamma L_\beta}{3\gamma^2}\right)\left(a_i b_j + a_j b_i\right) + \left(\rho_0 - L_\beta^2\right) b_i b_j = 0$, i.e. $h_{ij} \neq 0$

Hence the isotropic E-curvature $\bar{E}_{ij} = 0$.

Theorem 5.1. *Let $\bar{L} = e^c L$ be the conformal change of Lagrangian metric* L. Suppose $\bar{L}$ has isotropic mean Berwald curvature. Then it reduced to a weakly Berwald metric.

6. Conclusion

The article starts with the basic definition of cubic and β metrics with the formulation of Lagrange space with (γ,β)-metrics. The next part of the article gives a conformal change of Lagrangian metrics and locally dually flat change in Lagrange space. There are some results on isotropic E-curvature.

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