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Analytical Solution of an Advection Dispersion Equation for the Water Pollutant Concentration

Prem Sagar Bhandari

Department of Mathematics, Birendra Multiple Campus, Tribhuvan University, Nepal
Email: *premsagar61075[at]gmail.com*

Abstract: ***In this paper, an analytical solution of an advection- dispersion equation for the concentration of pollutant in one - dimension is derived. The Laplace transform technique has been used to solve the advection dispersion equation with taking added pollutant rate along the river zero. It is obtained that the concentration of the pollutants decreases continuously with increasing distance of the river for constant as well as different time and velocity at the origin.***

Keywords: Advection- dispersion, Analytical solution, Laplace transform technique, Concentration of pollutant.

1. Introduction

Water pollution is a big problem for the human beings and environment. Water pollution affects not only to the individual species and populations but also to the natural biological communities [17].

We people and all living species are facing bad effect of contaminated water [1].Water pollutants are biological waste materials that experience several biological and biodegradation steps using dissolved oxygen [10]. Several analytical and mathematical models have used to predict the concentration of water pollution.

Kumar et al. have given an analytical solution of one-dimensional advection equation in longitudinal semi-infinite homogeneous porous media applying Laplace transformation method [7]. Huang et al. obtained an analytical solution for one-dimensional transport in homogeneous porous media for scale dependent dispersion applying Laplace transformation technique [3].

There are several works in one -dimensional advection diffusion equation to describe the dispersion of water pollutants. Some of works have been collected [Rounds [12],Smith [15],Marine [8],Demuth [2], Yadav et al. [18], Jaiswal et al. [5], Kumar, A.et al.[6], Jaiswal.et.al.[4], Mourad, F.D. [9], Pimpunchat, et al. [11], Savoris, S. [13], Sirin, H. [14], Wagmare, R.V. and Kiwne, S.B.[16]].

In this paper, we solved the advection-diffusion equation using Laplace technique and examined the concentration profile of water pollutants with taking ‘p’ initial rate of pollutants and zero added pollutants rate along the river.

2. Mathematical Formulation and Solution:

An equation which describes an unsteady flow of water pollutant concentration in one- dimension can be given by a differential equation as Pimpunch et al. [9]:

$$\frac{\partial(AC)}{\partial t} = D_x \frac{\partial^2(AC)}{\partial t^2} - \frac{\partial(UAC)}{\partial x} - k_1 \frac{X}{X+k} AC + q;\ 0 \le x < L, \quad t > 0 \quad (1)$$

where U is the velocity of water in x- direction, A is the cross-section of area of river, $C(x,t)$ is the pollutant concentration, D_x is the coefficient of dispersion of pollutant in x- direction, q is the added pollutant rate along the river, k is the half-saturated oxygen demand concentration for pollutant decay, k_1 is the degradation rate of coefficient of pollutant and X is the dissolved oxygen concentration within the river. Wealso assume the stream reach is taken to be homogeneous system. So, we take the parameters A, U, D_x, k_1 as constants over time and space. We take $D_x = 0$ *and* $k = 0$.

For much greater pollutants D_x is approximately zero and so we take $D_x = 0$. Applying above mentioned conditions, the equation (1) becomes:

$$\frac{\partial(AC)}{\partial t} = -\frac{\partial(UAC)}{\partial x} - k_1 AC;\ 0 \le x < L, \quad t > 0 \quad (2)$$

Equation (2) is solved with the following conditions:

$$C(x,0) = p \ ; \qquad x \ge 0 \quad (3)$$

$$C(0,t) = r \ ; \qquad t > 0 \quad (4)$$

where the initial rate of pollution along the river is p and r is the rate of pollution at the origin.

Applying the Laplace transformation technique to (2) and (4); we have

$$s\overline{C}(x,s) - C(x,0) = -U\frac{\partial \overline{C}(x,s)}{\partial x} - k_1\overline{C}(x,s); s > 0 \quad (5)$$

$$\overline{C}(x,s) = \frac{r}{s} \quad (6)$$

where s is the Laplace transform variable.

Using equation (3) in equation (5); we have

$$s\overline{C}(x,s) - p = -U\frac{\partial \overline{C}(x,s)}{\partial x} - k_1\overline{C}(x,s);$$

On simplification, we get

$$\frac{\partial \overline{C}(x,s)}{\partial x} + \left(\frac{s+k_1}{U}\right)\overline{C}(x,s) = \frac{p}{U}; \quad s > 0 \quad (7)$$

which is a linear differential equation in $\overline{C}$,

so $I.F. = e^{\int \frac{k_1+s}{U} dx} == e^{\frac{k_1+s}{U}x}$

Using I.F. on (7) and integrating, we get

$$\overline{C}(x,s)e^{\frac{k_1+s}{U}x} = \int \frac{p}{U} e^{\frac{k_1+s}{U}x} dx$$

$$= \frac{p}{U}\int e^{\frac{k_1+s}{U}x} dx$$

On simplification, we get

$$\overline{C}(x,s) \ = \frac{p}{k_1+s} + c_1 e^{-\left(\frac{k_1+s}{U}\right)x} \qquad (8)$$

where c_1 is arbitrary constant.

Now, using the condition (6) to equation (8), we get

$$\frac{r}{s} = \frac{p}{k_1+s} + c_1 e^{-\left(\frac{k_1+s}{U}\right).0}$$

which gives

$$c_1 = \frac{r}{s} - \frac{p}{k_1+s}$$

Applying the value of c_1 in equation (8), we get

$$\overline{C}(x,s) \ = \frac{p}{k_1+s} + \left(\frac{r}{s} - \frac{p}{k_1+s}\right) e^{-\left(\frac{k_1+s}{U}\right)x}$$
$$= \frac{p}{k_1+s} + \left\{\frac{r}{s} - \frac{p}{k_1+s}\right\} e^{-\left(\frac{k_1+s}{U}\right)x}$$

Thus, we have

$$\overline{C}(x,s) = \frac{p}{(k_1+s)} + \frac{r}{s}\, e^{-\left(\frac{k_1+s}{U}\right)x} - \frac{p}{(k_1+s)} e^{-\left(\frac{k_1+s}{U}\right)x} \qquad (9)$$

Using inverse Laplace transform to equation (9), we have

$$C(x,t) = pe^{-k_1 t} + r\left\{e^{-\frac{k_1}{U}x}.H\left(t-\frac{x}{U}\right)\right\} - p\, e^{-\left(\frac{k_1}{U}x + k_1 t\right)}. H\left(t - \frac{x}{U}\right) \qquad (10)$$

where H is Heaviside function and we define

$H\left(t-\frac{x}{U}\right) = 1 \quad if\ t - \frac{x}{U} > 0;$.

$H\left(t-\frac{x}{U}\right) = 0 \quad if\ \ 0 > t - \frac{x}{U};$

For $t > \frac{x}{U}$, equation (10) becomes

$$C(x,t) = \ pe^{-k_1 t} + re^{-\frac{k_1}{U}x} \ - \ \ p\, e^{-\left(\frac{k_1}{U}x + k_1 t\right)} \qquad (11)$$

Now, applying the quantities:

$$x' = \frac{k_1}{U}x\,, \quad t' = k_1 t\,, \ \ p' = p, C'(x',t') = C(x,t)\,, \ r' = r$$

Equation (11) becomes:

$$C'(x',t') = p'e^{-t'} + r'e^{-x'} - p'e^{-(t'+x')} \qquad (12)$$

3. Results and Discussion

From the solution of an unsteady advection- dispersion equation (2), we studied the pollutant concentration $C'(x,t)$behavior. The concentration $C'(x,t)$is in kg/m^3 given by equation (12). The parametric values used in the analysis are taken as Pimpunchat et al. [8].
$t' = 6.616(t = 0.8\ day), 7.443(t = 0.9\ day), 8.27(t = 1\ day. k1 = 8.27\ per\ day,$
$A = 2100\ m^2\ , p' = 0.03,\ \ 0.05,\ \ 0.07\ in\ kg/m^3.$
$r' = 0.001,\ \ 0.002,\ \ \ \ 0.003\ \ in\ \ kg/m^3.$

To show the behavior of the concentration of pollutants profiles, we display the concentration distribution graphically under different conditions.

Figure 1 represents the concentration profile against the distance $(0 \leq x' \leq 5)$ for constant value of time$(t' = 6.616(t = 0.8\ day)$ and constant velocity p' at the origin. It is seen that as x' increases the value of $C'(x,t)$ decreases. It reaches a constant value near the sink. The effect of time is dominant near the upstream and very small near the downstream.

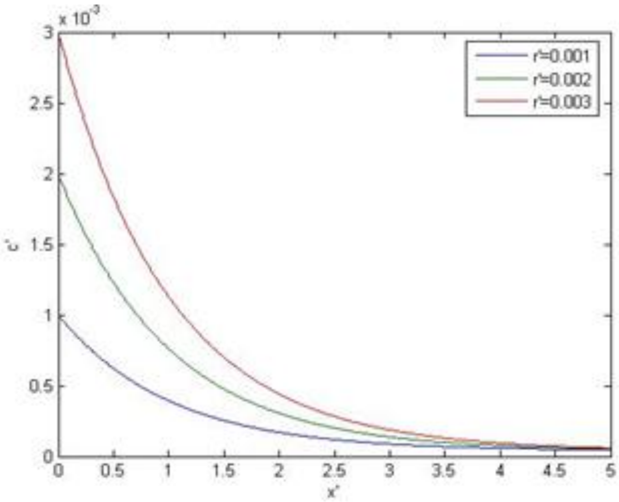

Figure 1: Variation of concentration for different r' with constant t'

Figure 2 represents the concentration profile against the distance $(0 \leq x' \leq 5)$ for different value of $p'\,and\ constant\ t'$.We found that as x' increases, the value of $C'(x,t)$ deceases for any time. The concentration $C'(x,t)$reaches a constant value near the sink.We seen that as initial velocity increases the value of concentration $C'(x,t)$ increases at any cross section of the river.

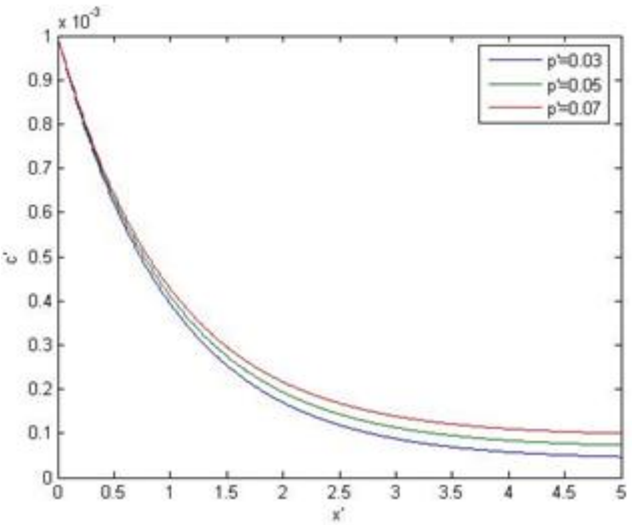

Figure 2: Variation of concentration for different p' and constant t'

Figure 3 represents the concentration profile against the distance $(0 \leq x' \leq 5)$ for different value of time and constant velocity at origin $r' = 0.001$ .It is seen that as x' increases, the concentration $C'(x,t)$ decreases for any time. We also found that the effect of time in concentration of pollutants is very small near the upstream and dominant near the downstream. We observed that as time increases the value of concentration $C'(x,t)$ increases at any cross section of the river.

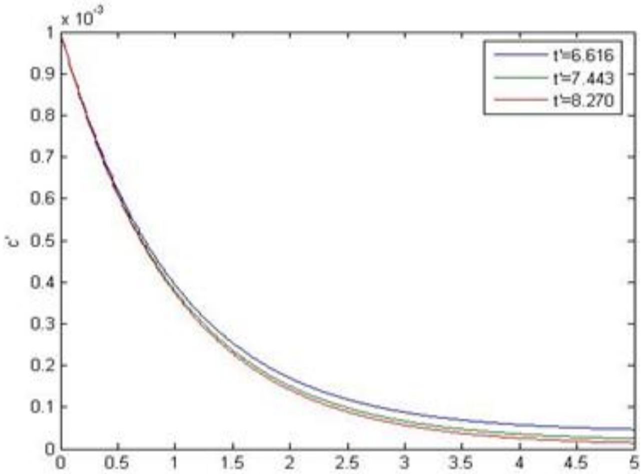

Figure 3: Variation of concentration for different t' and constant velocity

4. Conclusion

It is observed that the concentration profile of water pollutant is high near the source but as the distance from source increases, the concentration decreases continuously.

We have found that the concentration $C'(x,t)$ against the distance x' for different value of t' and constant velocity p' deceases for any time as x' increases. Minimum value of $C'(x,t)$ is seen near the downstream. As velocity at the origin increases, the value of concentration increases at any cross section of the river. The effect of time is dominant near the upstream and very small near the downstream.

This model is useful in studying the effect of time and velocity along the river and at the origin to predict the concentration of pollutants in the river

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