International Journal of Science and Research (IJSR)
ISSN: 2319-7064
SJIF (2019): 7.583

Application of Richardson's Arms Race Model in Oligopolistic Competition

Kevin Zhang[1], Zi Qian Chan[2]

[1, 2]United World College of South East Asia, East Campus, Singapore

Abstract: ***Richardson's Arms Race Model, originally proposed by mathematician Lewis Fry Richardson, has become an important tool in the study of arms races and international conflict. Its potential application outside of political and military science in economics for modeling competition in oligopolistic markets are illustrated.***

Keywords: competition, arms races, economics, mathematical modelling

1. Introduction

Arms races can be described as a situation wherein two or more countries increase their fighting resources and military capabilities to gain a position of superiority. In an attempt to model these complex events, Lewis Fry Richardson developed his arms race model: a pair of first-order ordinary differential equations aiming to capture the action-reaction nature of these situations [1]. While his model lacked specificity, it has nonetheless been influential in the advancements of conflict modeling [2]. What has been less explored, however, is the application of Richardson's ideas to other forms of conflict, such as amongst competing firms in an oligopolistic market. Given the core tenet of Richardson's model in presenting arms races as a scenario involving interconnected participants, this investigation aims to explore how such a model may be reinterpreted and reapplied onto other situations requiring analysis of interdependent stakeholders, specifically economics.

2. Richardson's Arms Race Model

Richardson's arms race model consists of a pair of first-order ordinary differential equations. First, assume two countries are engaged in an arms race where total military expenditure at time *t* for the two countries are *X(t)* and *Y(t)*, which are rewritten as *x* and *y* for simplicity [3]. Then, the rate the two countries increase or decrease their military spending at an instant of time can be expressed as the derivative of their spending with respect to time, or $\frac{dx}{dt}$ and $\frac{dy}{dt}$. According to Richardson, the value of these derivatives depends on three factors. Firstly, there's a positive 'defense' coefficient: increases in expenditure by one nation incentivizes the opposing nation to also increase expenditure to avoid inferiority. Secondly, there's a negative 'fatigue' coefficient: for each country, as their military expenditure levels increase, the rate for which they are further incentivized to spend is dampened. Finally, there's a 'grievance' constant which simply accounts for all other unconsidered details such as social perception. Richardson culminated these three factors to model the rate of change in military expenditure with respect to time for two opposing nations under an arms race [4].

$$\frac{dx}{dt} = ay - bx + c,$$

$$\frac{dy}{dt} = dx - ey + f,$$

$$a, d : positive\ 'defense'\ coefficients$$

$$b, e : negative\ 'fatigue'\ coefficients$$

$$c, f : 'grievance'\ constants$$

Figure 1: Richardson's arms race model

A primary weakness with Richardson's original model, however, is the challenge of finding the parameters through rigorous, quantitative methods. To address this, the model is often converted to a discrete-time model, allowing for easier parameterization [5]:

Let $X(n)$ be the level of armament of country 1, and $Y(n)$ be that of country 2. Assuming countries react to one another's armament levels:

$$X(n) = X(n-1) + a \cdot Y(n-1)$$

$$Y(n) = Y(n-1) + d \cdot X(n-1)$$

Given factors like budget cuts and reallocations, countries also have 'fatigue' to their level of armaments:

$$X(n) = X(n-1) - f_1 \cdot X(n-1) + a \cdot Y(n-1)$$

$$Y(n) = Y(n-1) - f_2 \cdot Y(n-1) + d \cdot X(n-1)$$

$$X(n) = a \cdot Y(n-1) + b \cdot X(n-1)$$

$$Y(n) = d \cdot X(n-1) + e \cdot Y(n-1)$$

To account for other factors and 'grievances', a constant is added for final models:

$$X(n) = a \cdot Y(n-1) + b \cdot X(n-1) + c$$

$$Y(n) = d \cdot X(n-1) + e \cdot Y(n-1) + f$$

Figure 2: Discrete-time version of Richardson's model

3. Potential Applications in Economics

While Richardson's arms race model was originally designed to model arms races, the principles of interdependence and action-reactionary processes can be reapplied and reinterpreted for use in economic modeling. One example of a potential application is the modeling of research and development (R&D) amongst competing firms in oligopolistic markets. Similar to nations in conflict, modern firms in oligopolistic markets face intense non-price competition and often strive to differentiate their products through R&D [6]. Assuming a duopoly, then $X(t)=x$ and $Y(t)=y$ can be reapplied as the research and development expenditures at a given time t. Then, the coefficients and constants to Richardson's original model and the modified discrete-time model can also be reinterpreted. In the context of research and development competition, these could serve as possible reinterpretations for the parameters in Richardson's original model:

a,d: 'Defense' coefficient of R&D: extent to which a firm feels motivated to increase R&D in response to heightened R&D of opposing firms. Fuelled by factors like the intensity of competition, degree of interdependence, the closeness of substitute.

b,e: 'Fatigue' coefficient of R&D: the extent to which one firm feels less incentivized to increase R&D given pre-existing expenditure levels. Fuelled by factors such as budgetary constraints or budget reallocations.

c,f: 'Grievance' constants of R&D: continuing to accounting for other unconsidered factors.

4. Apple Inc. and Samsung Electronics Co., Ltd. R&D Race

As an example application of Richardson's arms race model for R&D competition in oligopolistic markets, the discrete-time model version is applied to the R&D expenditures between Apple Inc. and Samsung Electronics Co., Ltd., two close competitors in markets such as smartphones and tablets. Although both companies have a diverse range of products that aren't fully comparable, for the purpose of illustrating the model's applicability and the limits of publicly available data, this will have to be accepted as a limitation to the research method.

4.1 Data Collection

The R&D expenditures were found from annual reports and financial statements publicly made available by Apple Inc. and Samsung Electronics Co., Ltd. for every fiscal year from 2007 to 2019. For Samsung's figures, the values were converted from South Korean won to the United States dollar using the December 31, 2019 exchange rate of
1 KRW=0.00087 USD.

R&D expenditures per year from 2007-2019, in billions of USD		
Year	Apple Inc.	Samsung Electronics Co., Ltd.
2007	0.78	5.284
2008	1.11	6.14
2009	1.33	6.87
2010	1.78	8.46
2011	2.43	9.26
2012	3.38	11.06
2013	4.48	13.75
2014	6.04	14.26
2015	8.07	13.81
2016	10.05	13.75
2017	11.58	15.62
2018	14.24	17.34
2019	16.22	17.50

Figure 3: R&D expenditure data. Collected from [7], [8], [9], [10]

4.2 Regression Analysis

Using the data outlined in *Figure 3*, multiple linear regression was completed to find the best fit discrete-time models where n is the number of years since 2006.

$$X(n) = 0.18 \cdot Y(n-1) + X(n-1) - 0.81$$

$$Y(n) = 0.06 \cdot X(n-1) + 0.89 \cdot Y(n-1) + 1.99$$

Figure 4: Parameterized discrete-time models describing R&D expenditures for Apple Inc. and Samsung Electronics Co., Ltd.

4.3 Solving for Stability

Given a discrete-time model, we may 'solve' the system for stability—that is, we attempt to find a state of the system (in this case, a certain level of annual R&D expenditure for each company) such that the system remains in this state for each additional time period. We demonstrate here two approaches to do so: first, by solving for a theoretical stable state and attempting to determine if and when the system would enter such a state [11]; second, by determining the eigenvalues of the transformation matrix of the system to assess the long term behaviour of the system and determine whether it can reach stability [5]. Both methods require the use of linear algebra.

First, we rewrite the discrete time model. We let

$$A_n = \begin{pmatrix} X(n) \\ Y(n) \end{pmatrix} \quad (1)$$

represent the state of the system at time *n*. We let

$$B = \begin{pmatrix} 1 & 0.18 \\ 0.06 & 0.89 \end{pmatrix} \quad (2)$$

be the transformation matrix of the system, representing the transformation of the system by the 'defense' and 'fatigue' coefficients. We let

$$C = \begin{pmatrix} -0.81 \\ 1.99 \end{pmatrix} \quad (3)$$

be the constant vector for the grievance constants.

Note that by this definition, we have that

$$BA_{n-1}+C$$
$$=\begin{pmatrix} 1\cdot X(n-1)+0.18\cdot Y(n-1)-0.81 \\ 0.06\cdot X(n-1)+0.89\cdot Y(n-1)+1.99 \end{pmatrix}$$
$$=\begin{pmatrix} X(n) \\ Y(n) \end{pmatrix}=A_n \quad (4)$$

and thus we may rewrite the system as

$$A_n=BA_{n-1}+C \quad (5)$$

Under the first approach, we first calculate a theoretical stable state and then determine if this system will ever reach that state. We conjecture that at a time *s*, the system enters the stable state A_s. By definition of a stable state, we know

$$A_s=BA_s+C \quad (6)$$

Quick algebraic manipulation gives us

$$A_s=(I-B)^{-1}C \quad (7)$$

where *I* represents the 2 by 2 identity matrix.

Applying the values we have, we quickly see that

$$A_s=(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 0.18 \\ 0.06 & 0.89 \end{pmatrix})^{-1}\begin{pmatrix} -0.81 \\ 1.99 \end{pmatrix}$$
$$=\begin{pmatrix} -24.9167 \\ 4.5 \end{pmatrix} \quad (8)$$

We note that the proposed steady state has a value of -24.9167 for *X(s)*, which is clearly nonsensical, as there is no way to achieve negative expenditure. Therefore, we may safely conclude without further calculation that this system cannot achieve a steady state.

Presuming we did not know this, we may attempt the second approach. As

$$A_n=BA_{n-1}+C \quad (9)$$

is a non homogeneous system, we cannot directly reduce into the form of eigenvalues and eigenvectors. Instead, we first consider the homogeneous part of the system which we denote

$$A'_n=BA'_{n-1}=B^nA_0 \quad (10)$$

where $A_0 = A'_0$ denotes the initial state of the system at time 0.

Let x_1 and x_2 be the eigenvectors of *B*, and λ_1 and λ_2 be their corresponding eigenvalues respectively. Because eigenvectors corresponding to different eigenvalues are linearly independent, we can say

$$A_0=c_1x_1+c_2x_2 \quad (11)$$

for some constants c_1 and c_2. By the definitions of eigenvectors and eigenvalues, we can then rewrite

$$A'_n=\lambda_1^nc_1x_1+\lambda_2^nc_2x_2 \quad (12)$$

for the homogeneous part of the system. We now correct this with a constant vector *D* such that

$$A_n=A'_n+D=B^nA_0+D \quad (13)$$

We note that

$$A_n=BA_{n-1}+C=B(B^{n-1}A_0+D)+C$$
$$=B^nA_0+BD+C \quad (14)$$

Combining equations, we get

$$B^nA_0+BD+C=B^nA_0+D \quad (15)$$

and working through the algebra yields

$$D=(I-B)^{-1}C \quad (16)$$

giving us the form for A_n:

$$A_n=\lambda_1^nc_1x_1+\lambda_2^nc_2x_2+(I-B)^{-1}C \quad (17)$$

which by calculation with our given values gives

$$A_n=1.06258^nc_1\begin{pmatrix} 0.944544 \\ 0.328385 \end{pmatrix}+0.82742^nc_2\begin{pmatrix} -0.721828 \\ 0.692072 \end{pmatrix}+\begin{pmatrix} -24.9167 \\ 4.5 \end{pmatrix} \quad (18)$$

Clearly, the eigenvalue with the largest absolute value, in this case 1.06258, will dominate as *n* increases. We denote this the *dominant* eigenvalue and use it to assess the long term behaviour of the system. Since 1.06258>1, it will increase without bound for large *n*, meaning there is no stable state for A_s.

5. Conclusion

From both methods, we conclude that this system cannot reach a stable state, and increases in R&D expenditure for both corporations will likely continue without bound. Given the trends demonstrated in the corporate histories for both Apple Inc. and Samsung Electronics Co., Ltd. as major competitors, the model aligns with current conventional economic wisdom that both corporations will continue to increase their R&D expenditure to remain competitive [12].

The result from this investigation demonstrates the potential for Richardson's Arms Race Model to be applied in competition situations within oligopolistic markets, and suggests that further work be conducted on applications of the model in other non-militaristic competitive situations, both economic and beyond. Possible specific areas for such further study include, for example, the exploration of the model's applicability for understanding interpersonal relationships within psychology.

References

[1] Richardson, L. F., &Rashevsky, N. (1960). Arms and insecurity: a mathematical study of the causes and origins of war. Stevens.
[2] Smith, R. P. (2019). The Influence of the Richardson Arms Race Model. Pioneers in Arts, Humanities, Science, Engineering, Practice Lewis Fry Richardson: His Intellectual Legacy and Influence in the Social Sciences. https://doi.org/10.1007/978-3-030-31589-4_3
[3] Sathasivam, K. (2017, August). Arms Race Modeling. Oxford Bibliographies. https://www.oxfordbibliographies.com/view/document/obo-9780199756223/obo-9780199756223-0231.xml.
[4] Karmeshu. (1980). Statistical Study of the Richardson's Arms-Race Model With Time Lag. Conflict Management and Peace Science, 5(1), 69–78. https://doi.org/10.1177/073889428000500105
[5] Fox, W. P. Arms Control and Warfare. https://www.westpoint.edu/sites/default/files/inline-images/academics/academic_departments/mathematical_sciences/Math/DDS1.pdf.
[6] Tishler, A., & Milstein, I. (2009). R&D wars and the effects of innovation on the success and survivability of firms in oligopoly markets. International Journal of Industrial Organization, 27(4), 519-531. https://doi.org/10.1016/j.ijindorg.2008.12.004
[7] EDGAR search results. (n.d.). SEC.gov. https://www.sec.gov/cgi-bin/browse-edgar?CIK=320193&owner=exclude
[8] Samsung Electronics: Research & development expenditure 2009-2019. (2020, January). Statista. https://www.statista.com/statistics/236924/samsung-electronics-research-and-development-expenditure/
[9] Samsung Electronics Co., Ltd. (2008). 2008 Samsung Electronics Annual Report. Samsung Electronics Co., Ltd. https://images.samsung.com/is/content/samsung/assets/global/ir/docs/SE2008_eng_final.pdf
[10] Samsung Electronics Co., Ltd. (2007). Samsung Electronics Annual Report. Samsung Electronics Co., Ltd. https://images.samsung.com/is/content/samsung/p5/uk/aboutsamsung/2017/pdf/about-us-sustainability-report-and-policy-annual-report-2007-en.pdf
[11] Hoffoss, D. (1999). Applied project #4 -- Fall 1999. University of San Diego. https://home.sandiego.edu/~dhoffoss/teaching/math320/project4.html
[12] Rivalry between Apple and Samsung in smartphones will grow fiercer. (2017, September 14). The Economist. https://www.economist.com/business/2017/09/14/rivalry-between-apple-and-samsung-in-smartphones-will-grow-fiercer