Marcinkiewicz-Zygmund Type Inequality on Arcs of the Circle for Generalized Polynomial

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Abstract

Let $1 < p < \infty$ and $0 \le \alpha < \beta \le 2\pi$. Let $\alpha = t_1 < t_2 < t_3 < \cdots < t_N = \beta$ and let $\Delta := \{e^{i\theta} : \theta \in [\alpha, \beta]\}$. For a generalized nonnegative algebraic polynomial P of (generalized) degree $\le N$, we prove

$$\sum_{j=1}^{N} |P|^p \left|t_j - t_{j-1}\right| \le C \int_{\alpha}^{\beta} |P|^p \, dt$$

with restricted zeros, where C is (a constant) independent of α, β, N and P and $t_1, t_2 \ldots t_N \in \Delta$.

Introduction

The large sieve of number theory may be viewed as an inequality for algebraic polynomials $P(z) = \sum_{j=0}^{n} d_j z^j$ on the unit circle T of the form

$$\sum_{j=1}^{m} \left|P\left(e^{i\alpha_j}\right)\right|^2 \le \left(\frac{n}{2\pi} + \frac{1}{\delta}\right) \int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^2 d\theta \qquad (1)$$

where $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_m \le 2\pi$ and $\delta := \min\{\alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \ldots\ \alpha_m - \alpha_{m-1}, 2\pi - (\alpha_m - \alpha_1)\ \} > 0$.

This particular form may be deduced from Theorem 3 in [9] by a substitution. The large sieve has been extended in numerous directions. For instance, $|P|^2$has been replaced by $|P|^p$ or, in more general form, by $\psi(|P|^p)$, where ψ is convex, nonnegative, and nondecreasing function. Moreover, polynomials have been replaced by generalized polynomials. In 1999, L Golinskii, D S Lubinsky, and P Nevai [3] established inequalities like (1) with integrals over arcs of the circle, rather than whole circle. They proved

Theorem 1 (LG, DSL, PN)

Let $0 < p < \infty$ and, $0 < n < \infty$ and assume that $0 \le \alpha < \beta \le 2\pi$. Consider the arc $\Delta = \Delta(\alpha, \beta) = \{e^{i\theta} : \theta \in [\alpha, \beta]\}$ and the quadratic polynomial R defined by

$R(z) \coloneqq (z - e^{i\alpha})(z - e^{i\beta})$. Let

$$\varepsilon(z) \coloneqq \frac{1}{pn+1}\left[|R(z)| + \left(\frac{\beta-\alpha}{pn+1}\right)^2\right]^{\frac{1}{2}}$$

and let $m \in N$. Assume that $a_j = e^{i\alpha_j} \in \Delta, 1 \le i \le m$.

Then for every generalized algebraic polynomial P of degree n, we have

$$\sum_{j=1}^{m} |P(a_j)|^p \varepsilon(a_j) \le C_\tau \int_\alpha^\beta |P(e^{i\theta})|^p d\theta,$$

where $\tau = \tau(\alpha, \beta, p, n, \{a_j\})$ is defined by

$$\tau \coloneqq \max_{\gamma \in [\alpha,\beta]} \left|\{j: \alpha_j \in [\gamma - \varepsilon(e^{i\gamma}), \gamma + \varepsilon(e^{i\gamma})]\}\right|$$

and $C \ne C(\alpha, \beta, p, m, , n, P, and \{a_j\})$ is an absolute constant.

This implies large sieve inequalities for generalized (n0negative) trigonometric polynomials of degree n on the subinterval of $[0, 2\pi]$. The essential feature is the uniformity of the estimate in α and β.

Our Results

We prove an inequality of this form which may be viewed as converse Marcinkiewicz-Zygmund type inequality on all arcs of the unit circle for a nonnegative, generalized algebraic polynomials. Our main result is

Theorem 2

Let $p > 1$. Let $0 \le \alpha < \beta \le 2\pi$ and $\alpha = t_0 < t_1 < \cdots < t_n = \beta$. Assume that $t_{j+1} - t_j \le K\varepsilon_N(t), \forall\, t \in [t_j, t_{j+1}]$, where K is an arbitrary constant, independent of $j, n,$ and t and

$$\varepsilon_N(z) :\!= \frac{1}{N}\left[\frac{(z - e^{i\alpha})(z - e^{i\beta}) + \left(\frac{\beta-\alpha}{N}\right)^2}{\left|z + e^{i\frac{\alpha+\beta}{2}}\right|^2 + \left(\frac{1}{N}\right)^2}\right]^{\frac{1}{2}}$$

Let $\Delta :\!= \{e^{i\theta}: \theta \in [\alpha, 2\pi - \alpha]\}$ and assume for all j,

$$z_j \notin \bigcup_{z\in\Delta} \left\{t: |t - z| \le \frac{\varepsilon_N(z)}{100}\right\} =: \gamma$$

Then for every generalized nonnegative algebraic polynomials of generalized degree N, we have

$$\sum_{j=1}^{N} |P|^p \left|t_j - t_{j-1}\right| \leq C \int_{\alpha}^{\beta} |P|^p \, dt$$

where C is a constant, independent of α, β, N, and P.

We prove the Theorem 2 using the following Theorem (Theorem 3) proved by the author in [6], Holder's Inequality and the Fundamental Theorem of Reimann Integration.

Theorem 3 (KK)

Let $P(z) \coloneqq \omega \prod_{j=1}^{n} \left(z - z_j\right)^{r_j}$ with $r_j \geq 1, r_j \in R$, and $\omega \in C$ be a nonnegative generalized algebraic polynomial with generalized degree $N \coloneqq \sum_{j=1}^{n} r_j$. Let $0 < p < \infty$ and let $0 \leq \alpha < \beta \leq 2\pi$. Let

$$\varepsilon_N(z) := \frac{1}{N} \left[\frac{\left(z - e^{i\alpha}\right)\left(z - e^{i\beta}\right) + \left(\frac{\beta - \alpha}{N}\right)^2}{\left|z + e^{i\frac{\alpha+\beta}{2}}\right|^2 + \left(\frac{1}{N}\right)^2} \right]^{\frac{1}{2}}$$

and let $\Delta := \left\{e^{i\theta} : \theta \in [\alpha, 2\pi - \alpha]\right\}$. Assume for all j,

$$z_j \notin \bigcup_{z \in \Delta} \left\{ t : |t - z| \leq \frac{\varepsilon_N(z)}{100} \right\} =: \gamma$$

Then we have

$$\int_{\alpha}^{\beta} \left|P' \varepsilon_N(e^{i\theta})\right|^p d\theta \leq C \int_{\alpha}^{\beta} \left|P(e^{i\theta})\right|^p d\theta,$$

where C is independent of α, β, N, and P.

Proof of our Result (Theorem 2)

Choose $s \in \left[t_j, t_{j+1}\right]$ such that

$$|P(s)|^p = \min_{[t_j, t_{j+1}]} |P|^p$$

From the Fundamental Theorem of Reimann Integrals

$$\left|P\left(t_j\right)\right|^p = |P(s)|^p + \int_{s}^{t_j} \frac{d(|P(t)|^p)}{dt} dt$$

$$\leq \min_{[t_j,t_{j+1}]}|P|^p + p\int_{t_j}^{t_{j+1}}|P|^{p-1}|P'|$$

Hence

$$|P(t_j)|^p(t_j - t_{j-1}) \leq (t_j - t_{j-1})\min_{[t_j,t_{j+1}]}|P|^p + p(t_j - t_{j-1})\int_{t_j}^{t_{j+1}}|P|^{p-1}|P'|$$
$$\leq \int_{t_j}^{t_{j+1}}|P|^p + p(t_j - t_{j-1})\int_{t_j}^{t_{j+1}}|P|^{p-1}|P'|$$

and thus

$$\sum_{j=0}^{n}|P(t_j)|^p(t_j - t_{j-1}) \leq \int_\alpha^\beta|P|^p + p\sum_{j=0}^{n}(t_j - t_{j-1})\int_{t_j}^{t_{j+1}}|P|^{p-1}|P'|$$

Now, the Theorem 3 and our assumption on the Theorem that $t_{j+1} - t_j \leq K\varepsilon_N(t), \forall\, t \in [t_j, t_{j+1}]$ together imply that

$$\sum_{j=0}^{n}|P(t_j)|^p(t_j - t_{j-1}) \leq \int_\alpha^\beta|P|^p + C_1\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}|P|^{p-1}|P'|\,\varepsilon_N$$
$$= \int_\alpha^\beta|P|^p + C\int_\alpha^\beta|P|^{p-1}|P'|\,\varepsilon_N$$
$$\leq \int_\alpha^\beta|P|^p + C_1\left(\int_\alpha^\beta|P|^{(P-1)\left(\frac{P}{P-1}\right)}\right)^{\frac{P-1}{p}}\left(\int_\alpha^\beta(|P'|\varepsilon_N)^P\right)^{\frac{1}{P}}$$
$$\leq \int_\alpha^\beta|P|^p + C_1\left(\int_\alpha^\beta|P|^p\right)^{\frac{P-1}{p}}\left(C_2\int_\alpha^\beta(|P|)^P\right)^{\frac{1}{P}}$$
$$= (1 + C_3)\int_\alpha^\beta|P|^P = C\int_\alpha^\beta|P|^P \qquad \blacksquare$$

Here, we have used Holder's Inequality.

Our estimate is uniform in all intervals $[\alpha, \beta]$ even as this approach $[0,2\pi]$, while the estimate of the Theorem 1 does not hold uniformly in such a case.

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