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A Remark on Special Class of Functions

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Abstract: ***A line closure function L(S) is defined as union of all lines connecting 2 distinct points A and B in S. L(S) order "n" is a function L(S) applied to a set S, n times. We explore the minimum amount of times we need to take a line closure of a set of n + 1 points in n space until the entire n-plane is covered.***

1. Introduction and Main Results

Lemma- Suppose a set S that satisfies this condition lies within n-space, but not all points are on a single n-1 dimensional hyperplane. I claim that S has to be an n dimensional hyperplane

Proof - First, consider a set K_n of n+1 points from S which are not all on the same n-1 plane. These points have to exist, since n points always lie on an n-1 plane, so if we couldn't find such a set, all points in S would be in n-1 space. Now, define the interative line closure $^n(S)$ to be (((...(S)..))), where is applied n times. I claim that $^{2n}(K_n)$ is the entire n-plane (note that this is a very loose bound.) However, since $^{2n}(K_n)$ ^{2n} $(S) = S$, and S is enclosed in the n-plane, it must be the entire n-plane, if this lemma is true.

So, we wish to show: Given a set K_n of n + 1 points not all in the same n- 1 plane, $^{2n}(K_n)$ is the entire n-plane which encloses it. We will show this result with induction. Clearly, the claim is true for $n = 1$, as $^2(K_1)$ is just the line connecting the two points in K_1. Now, suppose K_n works, and $K_{n+1} = K_n \cup P$, for some point P not on the same n-plane.

We wish to find $^2(^{2n}(K_n \cup P)) \sqsupseteq^2 (^{2n}(K_n) \cup P)$.

First, look at $(^{2n}(K_n) \cup P)$. For any point Q in the n + 1 plane determined by K_{n+1}, PQ will pass the hyperplane determined by K_n unless PQ is parallel to $L^{2n}(K_n)$. So, as long as PQ isn't parallel to the n-plane, there exists a point R on $^{2n}(K_n)$ such that PR goes through Q. This means $(^{2n}(K_n)\ P$) is just the desired n + 1 plane, with an n-plane going through P and parallel to $^{2n}(K_n)$ removed. (Denote this plane as P_p.)

Now, of course, we have nearly all the points in the desired n + 1 plane, so we can fill in the remaining points with a second line-closure. For completeness, consider two planes parallel to P_p, one above and one below it, and both of which are within $(^{2n}(K_n) \cup P)$. Taking a line passing through corresponding points of the two n-planes will pass through the corresponding point of P_p, so $^2(^{2n}(K_n) \cup P)$ contains every point in the n+1 plane determined by K_{n+1}. This line closure is a subset of $^{2(n+1)}(K_{n+1})$, however this iterated line closure does not exceed the n + 1 plane, so $^{2(n+1)}(K_{n+1})$ is also the n + 1 plane, as desired.

Now, we know that if the smallest dimension which contains a satisfactory set is n, then S is an n-plane. So, our solution set is precisely the set of n-planes as n ranges from 0 to infinity. It is obvious that they all are indeed their own line closures.

Now, we will extend our results to general n dimensional space. In particular, we will attempt to find $o(n)$, which denotes the minimum amount of times we need to take a line closure of a set of n + 1 points in n space until the entire n-plane is covered.

First, note that line closures take all possible linear combinations of points within a certain set. So, given a set of n points S, (S) contains linear combinations of 2 elements, then $^2(S)$ contains linear combinations of at most 4 elements. In general, $^k(S)$ has linear combinations composed of at most 2^k elements of S. Therefore, we can get the bound $o(n) \geq \log_2(n + 1)$, since there are some points which can only be expressed as linear combinations of all the points, such as the centroid. In fact, this bound is generally tight. But before we show anything, we will prove a lemma:

Lemma: Given a set of n + 1 points S, such that not all of them are on the same $n - 1$ plane, and two sets $A, B \subset S$ with $A \cup B = S$, $A \cap B \not= \emptyset$, define the hyperplane of dimension $|A| - 1$ determined by A to be P_A, and similarly define P_B. Then, $(P_A \cup P_B)$ is the n-plane determined by S.

Proof: Suppose the points in S are $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_k}$. Also WLOG the points in A are $\vec{x_1}, \ldots, \vec{x_m}$ and those in B are $\vec{x_{k-n+1}}, \ldots, \vec{x_k}$ (Of course, we can order the x suitably such that this is true.) Then, P_A is just the set of all linear combinations of the m points contained in A, and likewise for P_B. Now, given a linear combination of all k points, say $(v_1, v_2, \ldots, v_k)$ with $v_1 + v_2 + \ldots + v_k = 1$, (using similar notation as before), I claim that the line closure $(P_A \cup P_B)$ will contain it. Indeed, consider the linear combination $\frac{1}{v_1+\ldots+v_m}(v_1, v_2, \ldots, v_m)$ of the elements of A, and the linear combination $\frac{1}{v_{m+1}+\ldots+v_k}(0, 0, \ldots, 0, v_{m+1}, v_{m+2}, \ldots, v_k)$ of k elements of B. This point exists because $A \cap B \not= \emptyset$, so $n > k - m$. Now, these two points are in P_A, P_B respectively, so the line closure will include the point

$$(v_1, v_2, \ldots, v_m, 0, \ldots, 0) + (0, \ldots, 0, v_{m+1}, \ldots, v_k) = (v_1, v_2, \ldots, v_k)$$

as desired. The only way where this doesn't work is if $\upsilon_1 + \ldots + \upsilon_m$ or $\upsilon_{m+1} + \ldots + \upsilon_k$ is equal to 0. However, this can be easily rectified.

In the former case, we can instead choose $\frac{1}{v_1+\ldots+v_m+\epsilon}(v_1, v_2, \ldots, v_m + \epsilon)$ and $\frac{1}{v_{m+1}+\ldots+v_k-\epsilon}(0, 0, \ldots, 0, -\epsilon, v_{m+1}, v_{m+2}, \ldots, v_k)$ for some very small epsilon, and this will be a valid construction. (This is why it was very important for A, B to have an intersection!) The same idea works for the latter, so the proof is complete. Note that even if P_A and P_B have some point holes in them, (i.e. they are almost the entire plane, like in the solution we found for the previous part) this proof will still apply since if the construction falls on a hole, we can once again add small epsilons until it isn't. This will be useful later.

Now that the lemma is complete, we proceed to the main question. I claim that

$$o(n) = [\log_2(n + 2)]$$

for $n > 1$. Note that we originally got $o(n) \geq \log_2(n + 1)$, so this claim is actually the minimum we got for all numbers except $n = 2^k$-1. In the special case of $n = 2^k$ -1, I claim that if we take the line closure only k times, we will get nearly the entire 2^k -1 plane, but we will miss a few points (like in the $n = 3$ case.) For the proof we will use strong induction. We are already done with base cases $n = 2, 3$, so assume that the claim is true for all $n < k$.

First, consider a set of $k + 1$ points $S_k = (x_1, . . . , x_{k+1})$ such that not all of them are in the same $k - 1$ plane.

Suppose that $k \neq 2^a - 1$ for some a. If I take $^{[\log_2(n+2)]-1}(S_k)$, we will not get the entire k-plane by our inequality. However, if we let m be the largest power of 2 less than or equal to k, we know by inductive hypothesis that this line closure contains nearly the entire hyperplanes determined by $(x_1, . . . , x_m)$ and $(x_{k-m+2}, . . . , x_{k+1})$. We know that $k \leq 2(m\text{-}1)$ based on definition of m, so these two sets of m points have a nonzero intersection. So, by our lemma, $^{[\log_2(n+2)]-1}(S_k) = {}^{[\log_2(n+2)]}(S_k)$ is just the entire hyperplane, as desired.

On the other hand, the proof for $k = 2^a-1$ is very reminiscent of our derivation of $^2(\{A, B, C, D\})$. Basically, we will once again consider the two sets of points $\{x_1, . . . , x_{2^{a-1}}\}$ and $\{x_{2^{a-1}+1}, . . . , x_{2^a}\}$. These two sets determine two 2^{a-1}-1 hyperplanes, which are almost completely inside the image of $^{a-1}(S_k)$, by inductive hypothesis. Let these two hyperplanes be A and B respectively. We will now compute $(A \cup B)$. Using a similar vector based idea as before, we will get that all points $P = (v_1, v_2, . . . , v_{2^a})$ are in the line closure besides those on hyperplanes $v_1 + . . . + v_{2^{a-1}} = 0$ and $v_{2^{a-1}+1} + . . . + v_{2^a} = 0$. However, $(A \cup B)$ does not accurately represent a subset of $^a(S_k)$, since not the entire A or B was initially inside $^{a-1}(S_k)$. For example, suppose a point $Q = (q_1, . . . , q_{2^{a-1}})$, which is a linear combination of the points $x_1, . . . , x_{2^{a-1}}$ is in A but not in $^{a-1}(S_k)$.

Then, our vector solution tells us that any point $(v_1, v_2, . . . , v_{2^a})$ of the form

$$v_1 : v_2 : \cdots : v_{2^{a-1}} = q_1 : q_2 : \cdots : q_{2^{a-1}}$$

where the colons refer to ratio. Note that to construct a point of this form, we can first choose $v_{2^{a-1}+1}, . . . , v_{2^a}$, and then the remaining 2^{a-1} coordinates are determined. As we have 2^{a-1} degrees of freedom, this is a $2^{a-1} - 1$ plane. So, if we only consider the sets of points $\{x_1, . . . , x_{2^{a-1}}\}$ and $\{x_{2^{a-1}+1}, . . . , x_{2^a}\}$, we see that $^a(S_k)$ is the entire k-plane, minus $2k - 1$ planes and a finite amount of $2^{a-1}-1$ planes.

After this process is repeated for all 2^{a-1} sized subsets of S_k, the final line closure is the entire k-plane, minus a few points which are part of all generated hyper- planes over all subsets. This is because a set of k k-1 planes have intersection at most a point, and the 2^{a-1} 1 planes end up having negligible dimension. (To show these points actually do exist, consider $\frac{1}{2^a-2}(-1, 1, 1, \dots, 1)$.) So, we have shown that $^a(S_k)$ contains the entire hyperplane minus a few points, as desired.

The claim that $^{a+1}(S_k)$ is the entire hyperplane follows trivially.

References

[1] J. Cardinal, M. Payne, and N. Solomon, Ramsey-type theorems for lines in 3- space, Discrete Math. Theoret. Comput. Sci. 18 (2016).

[2] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, Discrete Comput. Geom. 5 (1990), 99–160. Problem Set 18 Ross Mathematics Program

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