

# $\psi$ -Primary Submodules

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**Abstract:** Let  $R$  be a commutative ring with identity and  $M$  be a unitary  $R$ -module. Let  $\delta(M)$  be the set of all submodules of  $M$ , and  $\psi: \delta(M) \rightarrow \delta(M) \cup \{\emptyset\}$  be a function. We say that a proper submodule  $P$  of  $M$  is  $\psi$ -primary if for each  $r \in R$  and  $x \in M$ , if  $rx \in P$ , then either  $x \in P + \psi(P)$  or  $r^n M \subseteq P + \psi(P)$  for some  $n \in \mathbb{Z}_+$ . Some of the properties of this concept will be investigated. Some characterizations of  $\psi$ -prime submodules will be given, and we show that under some assumptions prime submodules and  $\psi$ -primary submodules are coincide.

**Keywords:** Prime submodule, Primary submodules,  $\phi$ -prime submodules

## 1. Introduction

Throughout this paper,  $R$  is a commutative ring with identity and  $M$  is a unitary  $R$ -module. A proper ideal  $P$  of a ring  $R$  is primary if for all elements  $a, b \in R$ ,  $ab \in P$  implies either  $a \in P$  or  $b^n \in P$  for some  $n \in \mathbb{Z}_+$ , [1]. In the theory of rings, primary ideals play important roles. One of the natural generalizations of primary ideals which have attracted the interest of several authors in the last two decades is the notion of primary submodule. These have led to more information on the structure of the  $R$ -module  $M$ . For an ideal  $I$  of  $R$  and a submodule  $N$  of  $M$ , let  $\sqrt{I}$  denote the radical of  $I$ , and  $[N : M] = \{r \in R : rM \subseteq N\}$  which is clearly an ideal of  $R$ . A proper submodule  $P$  of  $M$  is called a primary submodule if  $r \in R$  and  $x \in M$  with  $rx \in P$  implies that  $r^n \in [P : M]$  for some  $n \in \mathbb{Z}_+$  or  $x \in P$ , [2]. A proper ideal  $I$  of  $R$  is said to be prime ideal if  $a \cdot b \in I$  implies that either  $a \in I$  or  $b \in I$ , [1]. A proper submodule  $N$  of  $M$  is said to be prime submodule of  $M$  if  $r \in R$  and  $x \in M$  with  $rx \in N$  gives that  $r \in [N : M]$  or  $x \in N$ , [3]. Khaksari and Jafari extended the notion of prime submodule to  $\phi$ -prime. Let  $M$  be an  $R$ -module and  $\delta(M)$  be the set of all submodules of  $M$  and  $\phi: \delta(M) \rightarrow \delta(M) \cup \{\emptyset\}$  be a function. A proper submodule  $P$  of  $M$  is said to be  $\phi$ -prime if  $r \in R$  and  $x \in M$ ,  $rx \in P \setminus \phi(P)$  implies that  $r \in [P : M]$  or  $x \in P$  [4]. In this paper, we define and study the notion of  $\psi$ -primary submodules. Let  $\delta(M)$  be the set of all submodules of  $M$  and  $\psi: \delta(M) \rightarrow \delta(M) \cup \{\emptyset\}$  be a function. A proper submodule  $P$  of  $M$  is said to be  $\psi$ -prime if for each  $r \in R$  and  $x \in M$ , if  $rx \in P$ , then either  $x \in P + \psi(P)$  or  $r^n M \subseteq P + \psi(P)$  for some  $n \in \mathbb{Z}_+$ .

## 2. Basic Properties of $\psi$ -Primary Submodules

First we give the following definition.

### Definition (2.1):

Let  $M$  be an  $R$ -module and  $\delta(M)$  be the set of all submodules of  $M$ . Let  $\psi: \delta(M) \rightarrow \delta(M) \cup \{\emptyset\}$  be a function. A proper submodule  $N$  of  $M$  is said to be  $\psi$ -primary if for each  $r \in R$  and  $x \in M$ , if  $rx \in N$ , then  $x \in N + \psi(N)$  or  $r^n M \subseteq N + \psi(N)$  for some  $n \in \mathbb{Z}_+$ .

### Remarks and Examples (2.2)

(1) It is clear that every primary submodule of an  $R$ -module  $M$  is  $\psi$ -primary submodule of  $M$ , but the convers is not true in general for example: Let  $M = \mathbb{Z}_{p^\infty}$  as a  $\mathbb{Z}$ -module,  $N = \langle \frac{1}{p^i} + \mathbb{Z} \rangle$  where  $p$  is a prime number. Then  $N$  is not primary submodule of  $M$ , since  $pk \cdot (1/p^{i+k} + \mathbb{Z}) = 1/p^i + \mathbb{Z} \in N$  for some  $k \in \mathbb{Z} +$ . But  $(1/p^{i+k} + \mathbb{Z}) \notin N$  and  $pk \notin [N : \mathbb{Z}_{p^\infty}] = 0$ . But  $N$  is  $\psi$ -primary submodule of  $M$ .

**Proof:** Let  $\psi: \delta(\mathbb{Z}_{p^\infty}) \rightarrow \delta(\mathbb{Z}_{p^\infty}) \cup \{\emptyset\}$ , where  $\psi(N) = \mathbb{Z}_{p^\infty}$ ,  $\forall N \subseteq M$ , then for each  $r \in \mathbb{Z}$ ,  $x \in \mathbb{Z}_{p^\infty}$ , if  $rx \in N$ , then  $x \in N + \psi(N) = \mathbb{Z}_{p^\infty}$ . Therefore  $N$  is a  $\psi$ -primary submodule of  $\mathbb{Z}_{p^\infty}$ .

(2) If  $\psi(N) \subseteq N$  or  $\psi(N) = 0$ , then every  $\psi$ -primary submodule of  $M$  is a primary submodule.

(3) Let  $N, W$  be two submodules of an  $R$ -module  $M$  and  $N \subseteq W$ . If  $N$  is  $\psi$ -primary submodule of  $M$  and  $\psi(N) \subseteq \psi'(N)$ , where  $\psi': \delta(W) \rightarrow \delta(W) \cup \{\emptyset\}$  and  $\psi: \delta(M) \rightarrow \delta(M) \cup \{\emptyset\}$ , then  $N$  is  $\psi'$ -primary submodule of  $W$ .

**Proof:** Let  $r \in R, m \in W$  such that  $rm \in N$ . Since  $N$  is  $\psi$ -primary submodule of  $M$ , so either  $m \in N + \psi(N)$  or  $r^n M \subseteq N + \psi(N)$  for some  $n \in \mathbb{Z}_+$ . But  $\psi(N) \subseteq \psi'(N)$

, so either  $m \in N + \psi'(N)$  or  $r^n M \subseteq N + \psi'(N)$  for some  $n \in \mathbb{Z}_+$ . Hence either  $m \in N + \psi'(N)$  or  $r^n W \subseteq N + \psi'(N)$  for some  $n \in \mathbb{Z}_+$ . Therefore  $N$  is  $\psi'$ -primary submodule of  $W$ .

(4) Given two function  $\psi, \psi': \delta(M) \rightarrow \delta(M) \cup \{\emptyset\}$ . We define  $\psi \leq \psi'$  if  $\psi(N) \subseteq \psi'(N)$  for each  $N \in \delta(M)$ . If  $N$  is a  $\psi$ -primary submodule of  $M$  implies  $N$  is  $\psi'$ -primary submodule of  $M$ .

**Proof:** Let  $r \in R, m \in M$  such that  $rm \in N$ . Since  $N$  is  $\psi$ -primary submodule of  $M$ , so either  $m \in N + \psi(N)$  or  $r^n M \subseteq N + \psi(N)$  for some  $n \in \mathbb{Z}_+$ . But  $\psi(N) \subseteq \psi'(N)$ . So either  $m \in N + \psi'(N)$  or  $r^n M \subseteq N + \psi'(N)$  for some  $n \in \mathbb{Z}_+$ . Therefore  $N$  is  $\psi'$ -primary submodule of  $M$ .

(5) Let  $N$  and  $W$  be two submodules of an  $R$ -module  $M$  such that  $N \cong W$ . If  $N$  is  $\psi$ -primary submodule of  $M$ ; it is not necessary that  $W$  is  $\psi$ -primary submodule of  $M$  as the following example explains:

Consider the  $Z$ -module  $Z$ , the submodule  $2Z$  is  $\psi$ -primary submodule of  $Z$  (since it is primary) but  $2Z \cong 30Z$  and  $30Z$  is not  $\psi$ -primary submodule of  $Z$ . Since  $\psi(N) = N, \forall N \subseteq M$  and  $6.5 = 30 \in 30Z$  but  $5 \notin 30Z + 30Z = 30Z$  and  $6Z \not\subseteq 30Z + 30Z = 30Z$ .

(6)  $I$  is a  $\psi$ -primary ideal of  $R$  if and only if  $I$  is a  $\psi$ -primary submodule of  $R$ .

(7) Let  $M = Z_{12}$  as a  $Z$ -module and  $N = \langle \bar{6} \rangle$ .  $N$  is not  $\psi$ -primary submodule of  $M$ .

**Proof:** Let  $\psi: \delta(Z_{12}) \rightarrow \delta(Z_{12}) \cup \{\emptyset\}$ , where  $\psi(N) = N + \langle \bar{6} \rangle, \forall N \subseteq Z_{12}$ . Now,  $2 \cdot \bar{3} = \bar{6} \in N$ , but  $\bar{3} \notin N + \psi(N) = N$  and  $2^n \notin [N + \psi(N): Z_{12}] = [N: Z_{12}] = [\langle \bar{6} \rangle: Z_{12}] = 6Z_{12}$  for each  $n \in \mathbb{Z}_+$ .

(8) The only  $\psi$ -primary submodule of a simple module is  $\{0\}$ . Therefore  $(\bar{0})$  of a simple  $Z$ -module  $Z_p$  ( $p$  is prime) is  $\psi$ -primary submodule.

(9) Let  $M = Z \oplus Z$  as a  $Z$ -module,  $N = 2Z \oplus (0)$ ,  $N$  is not  $\psi$ -primary submodule of  $M$ .

**Proof:** Let  $\psi: \delta(M) \rightarrow \delta(M) \cup \{\emptyset\}$  such that  $\psi(N) = N, \forall N \subseteq M$ . Now,  $2(1,0) \in N$ , but  $(1,0) \notin N + \psi(N)$  and  $2^n \notin [2Z \oplus (0): (Z \oplus Z)] = (0)$  for each  $n \in \mathbb{Z}_+$ .

Now, if  $N$  is a primary submodule, then sometimes  $N$  is called  $P$ -primary submodule, where  $P = \sqrt{[N: M]}$ , [5].

For a  $\psi$ -primary, we called  $P$ - $\psi$ -primary submodule, where  $P = \sqrt{[N + \psi(M): M]}$

The following theorem gives some characterizations for  $\psi$ -primary submodules.

**Theorem (2.3):**

Let  $N$  be a proper submodule of an  $R$ -module  $M$  and  $P = \sqrt{[N + \psi(M): M]}$

Then, the following statement are equivalent:

1.  $N$  is  $\psi$ -primary submodule of  $M$ .
2. For every submodule  $K$  of  $M$  and for every an ideal  $I$  of  $R$  such that  $IK \subseteq N$ , implies that either  $K \subseteq N + \psi(N)$  or  $I \subseteq P = \sqrt{[N + \psi(M): M]}$

**Proof:** (1)  $\rightarrow$  (2): Let  $IK \subseteq N$ , where  $I$  is an ideal of  $R$  and  $K$  is a submodule of  $M$ . Suppose  $K \not\subseteq N + \psi(N)$ , then there exists  $k \in K$  such that  $k \notin N + \psi(N)$ . It is clear that for each  $y \in I$ , thus  $yk \in N$ . But  $N$  is  $\psi$ -primary submodule of  $M$  and  $k \notin N + \psi(N)$ , hence  $y \in P = \sqrt{[N + \psi(M): M]}$ . Therefore  $I \subseteq P$ .

(2)  $\rightarrow$  (1): Let  $r \in R, m \in M$  such that  $rm \in N$ . Then  $\langle r \rangle \langle m \rangle \subseteq N$ . So either  $\langle m \rangle \subseteq N + \psi(N)$  or  $\langle r \rangle \subseteq P = \sqrt{[N + \psi(M): M]}$  by (2); i.e., either  $m \in N + \psi(N)$  or  $r \in P = \sqrt{[N + \psi(M): M]}$ . Therefore  $N$  is  $\psi$ -primary submodule of  $M$ .

We can give the following result.

**Proposition (2.4):**

Let  $N$  be a proper submodule of an  $R$ -module  $M$ . If  $\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): K]}$  for each submodule  $K$  of  $M$  such that  $K \supseteq N + \psi(N)$ , then  $N$  is  $\psi$ -primary submodule of  $M$ .

**Proof:** submodule of  $M$ . Let  $r \in R, m \in M$  such that  $rm \in N$  and suppose  $m \notin N + \psi(N)$ . Let  $K = N + \psi(N) + \langle m \rangle$ . Thus  $K \supseteq N + \psi(N), m \in K$  and so  $r \in [N: K] \subseteq [N + \psi(N): K] \subseteq \sqrt{[N + \psi(N): K]} = \sqrt{[N + \psi(M): M]}$ . It follows that  $r \in \sqrt{[N + \psi(M): M]}$  and hence  $N$  is  $\psi$ -primary.

However, we can give another corollary of proposition (2.4). But first we state and prove the following lemma which is needed.

**Lemma (2.5)**

Let  $N$  be a proper submodule of an  $R$ -module  $M$ . If  $\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): c]}$  for each  $c \in M \setminus N + \psi(N)$ ,

then  $\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): K]}$  for each submodule  $K$  of  $M$  such that  $K \supseteq N + \psi(N)$ .

**Proof:** Since

$K \subseteq M$  so  $\sqrt{[N + \psi(M): M]} \subseteq \sqrt{[N + \psi(M): K]}$ . Let  $r \in \sqrt{[N + \psi(M): K]}$ , hence  $r^n K \subseteq N + \psi(N)$  for some  $n \in \mathbb{Z}_+$ . But  $N + \psi(N) \subsetneq K$ , implies that there exists  $x \in K$  and  $x \notin N + \psi(N)$ . Hence  $r^n x \in N + \psi(N)$  for some  $n \in \mathbb{Z}_+$  and then  $r \in \sqrt{[N + \psi(M): x]} = \sqrt{[N + \psi(M): M]}$ , which implies that  $\sqrt{[N + \psi(M): K]} \subseteq \sqrt{[N + \psi(M): M]}$ .

Therefore  $\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): K]}$  for each submodule  $K$  of  $M$  such that  $K \supseteq N + \psi(N)$ .

**Corollary (2.6):**

Let  $N$  be a proper submodule of an  $R$ -module  $M$ . If  $\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): c]}$  for each  $c \in M \setminus N + \psi(N)$ , then  $N$  is  $\psi$ -primary submodule of  $M$ .

Now, the following proposition shows that under the condition  $\psi(N) \subseteq N$  for all submodule  $N$  of  $M$ . the convers of proposition (2.4) is true.

**Proposition (2.7):**

If  $N$  is a  $\psi$ -primary submodule of an  $R$ -module  $M$  and  $\psi(N) \subseteq N$ , then  $\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): K]}$  for each submodule  $K$  of  $M$  such that  $K \supseteq N + \psi(N)$ .

**Proof:**

Since  $N$  is a  $\psi$ -primary submodule of  $M$  and  $\psi(N) \subseteq N$ , so by (remark 2.2, (5))  $N$  is a primary submodule. Hence  $\sqrt{[N: M]} = \sqrt{[N: K]}$ , for each submodule  $K$  of  $M$  such that  $K \supseteq N$ , [6]. Since  $\psi(N) \subseteq N$ , then  $\sqrt{[N + \psi(M): M]} = \sqrt{[N + \psi(M): K]}$  for each submodule  $K$  of  $M$  such that  $K \supseteq N + \psi(N)$ .

It is well known if  $N$  is a primary submodule of an  $R$ -module, then  $[N: M]$  is a primary ideal of  $R$ , see [6]. But for a  $\psi$ -primary we have:

**Remark(2.8):**

If  $N$  is  $\psi$ -primary submodule of  $M$ , then it is not necessarily that  $[N: M]$  is a  $\psi$ -primary ideal of  $R$ .

Now, the following proposition shows that under the condition  $\psi(N) \subseteq N$  for all submodule  $N$  of  $M$ , the above statement is true.

**Proposition(2.9):**

If  $N$  is a  $\psi$ -primary submodule of an  $R$ -module  $M$  and  $\psi(N) \subseteq N$ , then  $[N: M]$  is a  $\psi$ -primary ideal of  $R$ .

**Proof:**

Since  $N$  is a  $\psi$ -primary submodule of an  $R$ -module  $M$  and  $\psi(N) \subseteq N$ , so  $N$  is a primary submodule by (2.2, 2), then  $[N: M]$  is a primary ideal of  $R$  and hence is a  $\psi$ -primary ideal of  $R$ .

**Remark (2.10):**

If  $[N: M]$  is  $\psi$ -primary ideal of  $R$ , then it is not necessarily that  $N$  is  $\psi$ -primary submodule of  $M$ , for example: Let  $M = Z \oplus Z$  as a  $Z$ -module,  $N = 2Z \oplus (0)$ ,  $N$  is not  $\psi$ -primary submodule of  $M$ , by (2.2, 8). But  $[N: M] = [2Z \oplus (0) : Z \oplus Z] = 0$  is a primary ideal of  $Z$  and hence is  $\psi$ -primary ideal of  $Z$ .

Now, we shall give characterization of  $\psi$ -primary submodules, but first recall the following: Let  $R$  be any ring. A subset  $S$  of  $R$  is called multiplicatively closed if  $1 \in S$  and  $ab \in S$  for every  $a, b \in S$ . We know that every proper ideal  $P$  in  $R$  is prime if and only if  $R/P$  is multiplicatively closed sub set of  $R$ , [1]. And if  $N$  is a submodule of an  $R$ -module  $M$  and  $S$  is multiplicatively closed sub set of  $R$ , then  $N(S) = \{x \in M : \exists t \in S, \text{ such that } tx \in N\}$  be a submodule of  $M$  and  $N \subseteq N(S)$ .

**Proposition (2.11):**

Let  $N$  be a proper submodule of an  $R$ -module  $M$ . If  $\sqrt{[N + \psi(M): M]}$  is a prime ideal of  $R$  and  $N(S) \subseteq N + \psi(N)$  for each multiplicatively closed sub set of  $R$  such that  $S \cap \sqrt{[N + \psi(M): M]} = \emptyset$ , then  $N$  is  $\psi$ -primary submodule of  $M$ .

**Proof:** Let  $r \in R, m \in M$  such that  $rm \in N$  and suppose  $m \notin N + \psi(N)$ ,  $r \notin \sqrt{[N + \psi(M): M]}$ . Claim the set  $S = \{1, r, r^2, \dots\}$ , this is multiplicatively closed sub set of  $R$  and it is clear that  $S \cap \sqrt{[N + \psi(M): M]} = \emptyset$ , since  $\sqrt{[N + \psi(M): M]}$  is a prime ideal of  $R$ . But  $m \notin N + \psi(N)$  implies that  $m \notin N(S)$  and so  $rm \notin N$  which is a contradiction. Therefore either  $m \in N + \psi(N)$  or  $r \in \sqrt{[N + \psi(M): M]}$  and hence  $N$  is  $\psi$ -primary submodule of  $M$ .

Conversely, if  $N$  is  $\psi$ -primary submodule of  $M$ , to prove  $N(S) \subseteq N + \psi(N)$ . Let  $x \in N(S)$ , so there exists  $t \in S$  such that  $tx \in N$ . But  $N$  is  $\psi$ -primary submodule of  $M$ , so either  $x \in N + \psi(N)$  or  $t \in \sqrt{[N + \psi(M): M]}$ . But

$t \in \sqrt{[N + \psi(M): M]}$  implies that  $S \cap \sqrt{[N + \psi(M): M]} = \emptyset$  which is a contradiction. Thus,  $x \in N + \psi(N)$  and hence  $N(S) \subseteq N + \psi(N)$ .

**Proposition (2.12):**

If  $\sqrt{[N + \psi(M): M]}$  is maximal ideal of  $R$ , then  $N$  is  $\psi$ -primary submodule of  $M$ .

**Proof:** Let  $r \in R, m \in M$  such that  $rm \in N$ . If  $r \notin \sqrt{[N + \psi(M): M]}$ , then  $R = \langle r \rangle + \sqrt{[N + \psi(M): M]}$ . Therefore there exist  $s \in R$  and  $k \in \sqrt{[N + \psi(M): M]}$  such that  $1 = sr + k$  and so  $m = srm + km \in N + \psi(N)$  for some  $n \in \mathbb{Z}_+$ . Therefore  $N$  is  $\psi$ -primary submodule of  $M$ .

**Proposition (2.13):**

Let  $N$  be a proper submodule of an  $R$ -module  $M$  such that  $[K: M] \not\subseteq [N + \psi(N): M]$  for each submodule  $K$  of  $M$  and containing  $N + \psi(N)$  properly. If  $[N + \psi(N): M]$  is a primary ideal of  $R$ , then  $N$  is  $\psi$ -primary submodule of  $M$ .

**Proof:** Suppose  $[N + \psi(N): M]$  is a primary ideal of  $R$ , to prove  $N$  is  $\psi$ -primary submodule of  $M$ . Let  $r \in R, m \in M$  such that  $rm \in N$  and suppose  $m \notin N + \psi(N)$ . Let  $K = N + \psi(N) + \langle m \rangle$ , it is clear that  $N + \psi(N) \subsetneq K$ , and so  $[K: M] \not\subseteq [N + \psi(N): M]$ . Then there exists  $s \in [K: M]$  and  $s \notin [N + \psi(N): M]$ . Thus,  $sM \subseteq K$  and  $sM \not\subseteq N + \psi(N)$ . But  $sM \subseteq K$  implies,  $rsM \subseteq rK = r(N + \psi(N) + \langle m \rangle) \subseteq N + \psi(N)$  and  $rs \in [N + \psi(N): M]$ . Since  $[N + \psi(N): M]$  is a primary ideal of  $R$  and  $s \notin [N + \psi(N): M]$ , so  $r^n \in [N + \psi(N): M]$  for some  $n \in \mathbb{Z}_+$ . Therefore  $N$  is  $\psi$ -primary submodule of  $M$ .

Recall that an  $R$ -module  $M$  is called multiplication module if for every submodule  $N$  of  $M$ , there exists an ideal  $I$  of  $R$  such that  $IM = N$ , equivalently; for every submodule  $N$  of  $M$ ,  $N = [N: M]M$ , see [7].

**Corollary (2.14):**

Let  $N$  be a proper submodule of a multiplication  $R$ -module  $M$ . Then  $N$  is  $\psi$ -primary submodule of  $M$  if  $[N + \psi(N): M]$  is a primary ideal of  $R$ .

**Proof:** Suppose  $[N + \psi(N): M]$  is a primary ideal of  $R$ , to prove  $N$  is  $\psi$ -primary submodule of  $M$ . Let  $r \in R, m \in M$  such that  $rm \in N$  and suppose  $m \notin N + \psi(N)$ . Let  $K = N + \psi(N) + \langle m \rangle$ , it is clear that  $N + \psi(N) \subsetneq K$ . Since  $M$  is multiplication, so  $[K: M] \not\subseteq [N + \psi(N): M]$  by [9, remark (2-15), chapter one]. Then there exists  $s \in [K: M]$  and  $s \notin [N + \psi(N): M]$ . Thus,  $sM \subseteq K$  and  $sM \not\subseteq N + \psi(N)$ . But,  $sM \subseteq K$  implies,  $rsM \subseteq rK = r(N + \psi(N) + \langle m \rangle) \subseteq N + \psi(N)$  and  $rs \in [N + \psi(N): M]$ . Since  $[N + \psi(N): M]$  is a primary ideal of  $R$  and  $s \notin [N + \psi(N): M]$ , so  $r^n \in [N + \psi(N): M]$  for some  $n \in \mathbb{Z}_+$ . Therefore  $N$  is  $\psi$ -primary submodule of  $M$ .

As another consequence of (2.13), we have the following result:

**Corollary (2.15):**

Let  $N$  be a proper submodule of a cyclic  $R$ -module  $M$ . Then  $N$  is  $\psi$ -primary submodule of  $M$  if  $[N + \psi(N):M]$  is a primary ideal of  $R$ .

**Proof:**

Since  $M$  is cyclic, then  $M$  is a multiplication. Hence the result follows immediately from corollary (2.14).

Recall that an  $R$ -module  $M$  is said to be a bounded module if there exists an element  $x \in M$  such that  $ann_R M = ann_R(x)$ , where  $ann_R M = \{r \in R: rm = 0, \forall m \in M\}$ , [8]. And an  $R$ -module is said to fully stable if each submodule is stable, where a submodule  $N$  of an  $R$ -module  $M$  is said to be stable if  $f(N) \subseteq N$  for each  $f \in Hom(N, M)$ , [9].

**Corollary (2.16):**

Let  $N$  be a proper submodule of a bounded fully stable  $R$ -module  $M$ . Then  $N$  is  $\psi$ -primary submodule of  $M$  if  $[N + \psi(N):M]$  is a primary ideal of  $R$ .

**Proof:**

Since  $M$  is a bounded fully stable  $R$ -module  $M$ , so  $M$  is a cyclic by [10]. Hence the result follows immediately from corollary (2.14).

**proposition (2.17):**

Let  $P$  be an ideal of a ring  $R$  and let  $M$  be an  $R$ -module. Then a proper submodule  $N$  of  $M$  is a  $P$ - $\psi$ -Primary if and only if

1.  $\subseteq \sqrt{[N + \psi(M):M]}$ , and
2.  $\not\subseteq N$ , for all  $c \in R \setminus P, m \in M \setminus N + \psi(N)$ .

**Proof:**

Suppose  $N$  is a  $P$ - $\psi$ -Primary. To prove that (1) and (2) are hold. It is clear that  $P = \sqrt{[N + \psi(M):M]}$ . Therefore  $\subseteq \sqrt{[N + \psi(M):M]}$ .

Now if  $c \in R \setminus P$  and  $m \in M \setminus N + \psi(N)$ , then  $c \notin \sqrt{[N + \psi(M):M]}$  and  $m \notin N + \psi(N)$ , hence  $cm \notin N$ . Conversely, let  $c \in R$  and  $m \in M$  such that  $m \notin N + \psi(N)$  and  $c \notin \sqrt{[N + \psi(M):M]}$ . Since  $\subseteq \sqrt{[N + \psi(M):M]}$ , then  $m \in M \setminus N + \psi(N)$  and  $c \notin P$ . Therefore,  $c \in R \setminus P$ . Hence  $cm \notin N$ , which implies that  $N$  is a  $P$ - $\psi$ -Primary.

**proposition (2.18):**

Let  $M$  be an  $R$ -module and  $N, L$  be two submodules of  $M$ . If  $K$  be a  $P$ - $\psi$ -primary submodule of  $M$  such that  $N \cap L \subseteq K$ , then  $L \subseteq K + \psi(K)$  or  $[N:M] \subseteq P = \sqrt{[K + \psi(M):M]}$

**Proof:**

Suppose  $[N:M] \not\subseteq \sqrt{[K + \psi(M):M]} = P$ , so there exists  $s \in [N:M]$  and  $s \notin P = \sqrt{[K + \psi(M):M]}$ . Let  $t \in L$ , then  $st \in L \cap N$  and so  $st \in K$ . But  $K$  is  $\psi$ -primary submodule of  $M$  and  $s \notin \sqrt{[K + \psi(M):M]}$ . Therefore  $t \in K + \psi(K)$ , thus  $L \subseteq K + \psi(K)$ .

**Corollary (2.19):**

Let  $A$  an ideal of  $R$  and  $N$  be a submodule of  $M$ . If  $K$  be a  $P$ - $\psi$ -primary submodule of  $M$  such that  $AM \cap N \subseteq K$ , then either  $AM \subseteq K + \psi(K)$  or  $N \subseteq K + \psi(K)$ .

**proposition (2.20):**

Let  $M$  be an  $R$ -module and  $N$  be a submodule of  $M$ . If  $P = [N + \psi(N):M]$  is a primary ideal of  $R$ , then  $\sqrt{[N + \psi(M):M]} = \sqrt{[N + \psi(M):rM]}, \forall r \notin \sqrt{[N + \psi(M):M]}$ .

**Proof:** Since  $rM \subseteq M$ ,

so  $\sqrt{[N + \psi(M):M]} \subseteq \sqrt{[N + \psi(M):rM]}$ . Let  $a \in \sqrt{[N + \psi(M):rM]}$ . Hence  $a^n \in [N + \psi(N):M]$  for some  $n \in \mathbb{Z}_+$ , and so  $a^n r \in N + \psi(N)$  which means that  $a^n r \in [N + \psi(N):M]$ . But  $[N + \psi(N):M]$  is a primary of  $R$  and  $r \notin [N + \psi(N):M]$ , so  $(a^{nm}) \in [N + \psi(N):M]$  for some. Thus,  $a \in \sqrt{[N + \psi(M):M]}$ . Therefore,  $\sqrt{[N + \psi(M):rM]} \subseteq \sqrt{[N + \psi(M):M]}$  and hence  $\sqrt{[N + \psi(M):M]} = \sqrt{[N + \psi(M):rM]}$ . Now, we can give the following proposition:

**Proposition (2.21):**

Let  $N$  be a submodule of an  $R$ -module  $M$  and  $P = \sqrt{[N + \psi(N):M]}$ . If the ideal  $\sqrt{[N + \psi(N):\langle e \rangle]} = P$ , for each  $e \in M, e \notin N + \psi(N)$ , then  $N$  is a  $\psi$ -primary submodule of  $M$ .

**Proof:** Let  $r \in R, x \in M$  such that  $rx \in N$  and suppose  $x \notin N + \psi(N)$ . Thus,

$r \in \sqrt{[N + \psi(N):\langle x \rangle]} = P$ . But  $[N + \psi(N):\langle x \rangle] = P$ , so  $r \in P$ . Therefore  $N$  is a  $\psi$ -primary submodule of  $M$ .

Note that, the intersection of two  $\psi$ -primary submodules of an  $R$ -module  $M$  need not be a  $\psi$ -primary submodule of  $M$ , for examples:

(1) The  $Z$ -module  $Z_6$  has two  $\psi$ -primary submodules,  $N_1 = \langle \bar{2} \rangle$  and  $N_2 = \langle \bar{3} \rangle$  but  $N_1 \cap N_2 = \langle \bar{0} \rangle$  is not a  $\psi$ -primary submodule of  $N_6$ . Since if  $r = 3, x = \bar{2}$  and  $\psi(N) = N \forall N \subseteq M$ , then  $rx = 3 \cdot \bar{2} = \bar{0} \in \langle \bar{0} \rangle + \psi(\langle \bar{0} \rangle) = \langle \bar{0} \rangle$ . But  $\bar{2} \notin \langle \bar{0} \rangle + \psi(\langle \bar{0} \rangle) = \langle \bar{0} \rangle$  and  $3 \notin \sqrt{[\langle \bar{0} \rangle + \psi(\langle \bar{0} \rangle):Z_6]} = \langle \bar{0} \rangle$ .

(2) The  $Z$ -module  $Z_{12}$  has two  $\psi$ -primary submodules,  $N_1 = \langle \bar{2} \rangle$  and  $N_2 = \langle \bar{3} \rangle$ . But  $N_1 \cap N_2 = \langle \bar{6} \rangle$  is not a  $\psi$ -primary submodule of  $Z_{12}$  as we have seen in (2.2, (7)). However, we have the following proposition:

**Proposition (2.22):**

Let  $K$  is a  $\psi$ -primary of an  $R$ -module  $M$  and let  $N < M$  such that  $\psi(K) \subseteq K$ . Then either  $N \subseteq K$  or  $K \cap N$  is  $\psi'$ -primary in  $N$ , where  $\psi': \delta(N) \longrightarrow \delta(N) \cup \{\phi\}$  and  $\psi: \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ .

**Proof:** Suppose that  $N \not\subseteq K$ , then  $K \cap N$  is a proper submodule in  $N$ . Let  $r \in R, m \in N$  such that  $rm \in K \cap N$ . Suppose  $m \notin (K \cap N) + \psi'(K \cap N)$ , where

$\psi': \delta(N) \longrightarrow \delta(N) \cup \{\phi\}$  be a function, then  $m \notin K$ . We must show that  $r^n N \subseteq (K \cap N) + \psi'(K \cap N)$  for some  $n \in \mathbb{Z}_+$ .

Since  $K$  is a  $\psi$ -primary submodule of  $M$  and  $m \notin K + \psi(K)$ , this implies that  $r^n M \subseteq K + \psi(K) = K$  for some  $n \in \mathbb{Z}_+$  and  $r^n N \subseteq K \subseteq K \cap N \subseteq K \cap N + \psi'(K \cap N)$  for some  $n \in \mathbb{Z}_+$ . Therefore  $K \cap N$  is a  $\psi'$ -primary in  $N$ .

**proposition (2.23):**

Let  $\phi: M \longrightarrow M'$  be an homomorphism. If  $N$  is  $\psi'$ -primary submodule of an  $R$ -module  $M'$ , such that  $\phi(M) \not\subseteq N$  and  $\psi(\phi^{-1}(N)) = \phi^{-1}(\psi'(N))$ , then  $\phi^{-1}(N)$  is  $\psi$ -primary submodule of  $M$ , where  $\psi': \delta(M') \longrightarrow \delta(M') \cup \{\phi\}$  and  $\psi: \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ .

**Proof:** First, we must show that  $\phi^{-1}(N)$  is a proper submodule of  $M$ . Suppose that  $\phi^{-1}(N) = M$ , then  $\phi(M) \subseteq N$ , which a contradiction to the assumption. Let  $r \in R, m \in M$  such that  $rm \in \phi^{-1}(N)$ . Then  $r\phi(m) \in N$  and as  $N$  is  $\psi'$ -primary submodule of an  $R$ -module  $M'$ , then either  $\phi(m) \in N + \psi'(N)$  or  $r^n M' \subseteq N + \psi'(N)$  for some  $n \in \mathbb{Z}_+$ . If  $\phi(m) \in N + \psi'(N)$ , then  $m \in \phi^{-1}(N) + \phi^{-1}(\psi'(N))$  and hence  $m \in \phi^{-1}(N) + \psi(\phi^{-1}(N))$ . If  $r^n M' \subseteq N + \psi'(N)$ , then  $r^n \phi(M) \subseteq N + \psi'(N)$  since  $\phi(M) \subseteq M'$ . This implies  $r^n M \subseteq \phi^{-1}(N) + \phi^{-1}(\psi'(N)) = \phi^{-1}(N) + \psi(\phi^{-1}(N))$  for some  $n \in \mathbb{Z}_+$ . Therefore  $\phi^{-1}(N)$  is  $\psi$ -primary submodule of  $M$ .

**Theorem (2.24):**

Let  $f: M \longrightarrow M'$  be an epimorphism and let  $N < M$  such that  $\ker f \subseteq N$ . If  $N$  is a  $\psi$ -primary submodule of a module  $M$  and  $\psi'(f(N)) = f(\psi(N))$ , then  $f(N)$  is a  $\psi'$ -primary submodule of a module of  $M'$ , where  $\psi': \delta(M') \longrightarrow \delta(M') \cup \{\phi\}$  and  $\psi: \delta(M) \longrightarrow \delta(M) \cup \{\phi\}$ .

**Proof:** First, we must show that  $f(N)$  is a proper submodule of a module  $M'$ . Suppose  $f(N) = M'$ . But  $f$  is an epimorphism, thus  $f(N) = f(M)$  and hence  $M = N + \ker f$ . This implies that  $M = N$ . A contradiction.

Now, let  $rm' \in f(N)$ , where  $r \in R$  and  $m' \in M', m' = f(m)$  for some  $m \in M$  since  $f$  is an epimorphism. Then  $rf(m) \in f(N)$ , so  $f(rm) = f(n)$  for some  $n \in N$  and hence  $f(rm) - f(n) = 0$ . Thus we get that  $rm - n \in \ker f \subseteq N$  which implies that  $rm \in N$ . But  $N$  is a  $\psi$ -primary, so either  $m \in N + \psi(N)$  or  $r^n M \subseteq N + \psi(N)$  for some  $n \in \mathbb{Z}_+$ . If  $m \in N + \psi(N)$ , then  $f(m) \in f(N) + f(\psi(N))$ ; that is  $m' \in f(N) + f(\psi(N)) = f(N) + \psi'(f(N))$ . If  $r^n M \subseteq N + \psi(N)$ , then  $r^n f(M) \subseteq f(N) + f(\psi(N)) = f(N) + \psi'(f(N))$  implies that  $r^n M' \subseteq f(N) + \psi'(f(N))$  for some  $n \in \mathbb{Z}_+$ .

**Corollary (2.25)**

Let  $M$  be an  $R$ -module, let  $K < N < M$  and  $N$  is a  $\psi$ -primary of  $M$ . Then  $N/K$  is a  $\psi'$ -primary submodule of  $M/K$  where  $\psi': \delta(M/K) \longrightarrow \delta(M/K) \cup \{\phi\}$ .

**Proof:** Let  $\pi: M \longrightarrow M/K$  be the natural mapping, then the result follows by proposition(2.25).

**Proposition (2.26):**

Let  $M$  be an  $R$ -module and let  $K < N < M$ . If  $N$  is a  $\psi$ -primary submodule of  $M$ , then  $N/K$  is a  $\psi'$ -primary submodule of  $M/K$  and  $\psi'(N/K) = \psi(N)/K$ .

**Proof:** Let  $r \in R$  and  $m \in N/K$  with  $rm \in N/K$ , where  $m = m + K$ , for some  $m \in M$ . So we have  $rm \in N$ , which gives that either  $m \in N + \psi(N)$  or  $rM \subseteq N + \psi(N)$ . Therefore either  $m + K \in (N + \psi(N))/K = N/K + \psi(N)/K = N/K + \psi'(N/K)$  or  $r^n M/K \subseteq (N + \psi(N))/K \subseteq N/K + \psi(N)/K = N/K + \psi'(N/K)$  for some  $n \in \mathbb{Z}_+$ . Hence either  $m \in N/K + \psi'(N/K)$  or  $r^n M/K \subseteq N/K + \psi'(N/K)$  for some  $n \in \mathbb{Z}_+$ . Therefore  $N/K$  is a  $\psi'$ -primary submodule of  $M/K$ .

Let  $S$  be a multiplicatively close subset of  $R$  and let  $R_s$  be the set of all fractional  $r/s$  where  $r \in R$  and  $s \in S$  and  $M_s$  be the set of all fractional  $x/s$  where  $x \in M$  and  $s \in S$ . For  $x, x \in M$  and  $s, s \in S, x/s = x/s$  if and only if there exists  $t \in S$  such that  $t(sx - sx) = 0$ . So, we can make  $M_s$  in to  $R_s$ -module by setting  $x/s + y/t = (tx + sy)/st$  and  $r/t \cdot x/s = rx/ts$  for every  $x, y \in M$  and  $s, t \in S, r \in R$ . And  $M_s$  is the module of fractions. If  $N$  be a submodule of an  $R$ -module  $M$  and  $S$  be a multiplicatively close subset of  $R$  so  $N_s = \{n/s : n \in N, r \in S\}$  be a submodule of the  $R_s$ -module  $M_s$ , [1, p.69].

Now, we state and prove the following proposition:

**Proposition (2.27):**

Let  $M$  be an  $R$ -module and  $N$  is a  $\psi$ -primary of  $M$ . Then  $N_s$  is a  $\psi_s$ -primary submodule of  $M_s$ , where  $\psi_s: \delta(M_s) \longrightarrow \delta(M_s) \cup \{\phi\}$  and  $[N + \psi(N)]_s = [N_s + \psi_s(N_s)]$  and  $[\psi(N)]_s = \psi_s(N_s)$ .

**Proof:** Let  $a/s \in R_s$  and  $x/t \in M_s$  with  $ax/st \in N_s$ . Then  $ax/st = n/u$  for some  $n \in N, u \in S$  and so  $axu = nst$  there exists  $u \in S$  such that  $uax \in N$ . Since  $N$  is  $\psi$ -primary of  $M$ . Then either  $x \in N + \psi(N)$  or  $(ua)^n \in [N + \psi(N):M]$  for some  $n \in \mathbb{Z}_+$ . Therefore either  $x/t \in [N + \psi(N)]_s = [N_s + \psi_s(N_s)]$  or  $a^n/s \in N_s + \psi_s(N_s):M_s \subseteq N_s + \psi_s(N_s):M_s = [N_s + \psi_s(N_s):M_s]$ . Therefore  $N_s$  is a  $\psi_s$ -prime submodule of  $M_s$ .

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