

# A Finite Capacity Single Server Markovian Queueing System with Discouraged Arrivals and Retention of Reneged Customers with Controllable Arrival Rates

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**Abstract:** In this Paper, a finite capacity, finite source single server queueing model with discouraged Arrivals and Retention of Reneged Customers Controllable arrival rates is considered. The steady state solutions of system size are derived explicitly. The analytical results are numerically illustrated and relevant conclusions are presented.

**Keywords:** Single Server, Finite Capacity, Finite Source, Customers Retention, Reneging, Discouraged Arrivals, Steady State Solution, Controllable Arrival Rates, Bivariate Poisson Process.

## 1. Introduction

Queues with discouraged arrivals have applications in Computer with batch job processing where job submissions are discouraged when the system is used frequently and arrivals are modeled as a Poisson process with state dependent arrival rate. Morse(1968) considers discouragement in which the arrival rate falls according to a negative exponential law. We consider the finite source of N customers arrive in a Poisson process with two arrival rates,  $\lambda_0, \lambda_1$  a faster and slower rate of arrivals which controls the arrivals. The discouragement affect the arrival rate to the queueing system. Customer arrive in a poisson fashion with rate depends on the number of customers present in the system at that time that is  $\frac{\lambda}{n+1}$ . The service times and reneging times follow exponential distribution with parameters  $\mu$  and  $\xi$ .

It is also assumed that whenever the queue size reaches a prescribed number R, the arrival rate reduces from  $\lambda_0$  to  $\lambda_1$  and it continues with the rate as long as the content in the queue was greater than some prescribed integer r [ $r \geq 0$  and  $r < R$ ] when the content reached r, the arrival rate changed back to  $\lambda_0$  and the same process is repeated. It is also assumed that there is finite source of N customers.

Queueing with impatience finds its origin during the early 1950's Haight(1959) studies a single server Markovian queueing system with reneging. Srinivaso Rao et al [5] have discussed M/M/1/ $\infty$  interdependent queueing model with controllable arrival rates. A. Srinivasan and M. Thiagarajan [6, 7] have analysed M/M/1/K interdependent queueing model with controllable arrival rates balking, reneging and spares.

Choudhury and Medhi [1] have studied customer impatience in multi server queues, Kapodistria [2] has studied a single server Markovian queue with impatient customers and considered the situations where customers abandon the system simultaneously. Kumar and Sharma [3] have studied M/M/1/N queueing system with retention of reneged customers. Recently S. Premalatha and M. Thiagarajan [4] have studied interdependent discouraged Arrivals and Retention of Reneged customers with Controllable arrival rates. An attempt is made in this paper to obtain the relevant results of the M/M/1/K/N interdependent discouraged arrivals and Retention of Reneged customers with Controllable arrival rates is considered.

## 2. Description of the Model

It is assumed that the arrival process  $[X_1(t)]$  and the service process  $[X_2(t)]$  of the systems are correlated and follows a bivariate poisson process is given by

$$P(X_1=x_1, X_2=x_2; t) = e^{-(\lambda_1 + \mu - \epsilon)t} \sum_{j=0}^{\min(x_1, x_2)} (\epsilon t)^d [(\lambda_1 - \epsilon)t]^{x_1-d} [(\mu - \epsilon)t]^{x_2-d}$$

Where  $x_1, x_2 = 0, 1, 2, \dots$

$$0 < \lambda_1, \mu$$

$$0 < \epsilon < \min(\lambda_1, \mu), i=0,1$$

with parameters  $\lambda_0, \lambda_1, \mu$  and  $\epsilon$  as mean faster and slower rate of arrivals, mean service rate and mean dependence rate (Co-variance between primary arrival and service processes) respectively.

## 3. Steady State Equations

Let  $P_n(0)$  denote the steady state probability that there are n customers in the system when the system is in the faster rate of arrival. Let  $P_n(1)$  denote the steady state probability that

there are n customers in the system when the system is in the slower rate of arrival.

We observe that only  $P_n(0)$  exists when  $n = 0, 1, 2, 3, \dots, r-1, r$ ; both  $P_n(0)$  and  $P_n(1)$  exist when  $n = R, R+1, \dots, K$ . Further  $P_n(0) = P_n(1) = 0$  if  $n > k$ .

The steady state equations are

$$-N(\lambda_0 - \epsilon)P_0(0) + (\mu - \epsilon)P_1(0) = 0 \quad \text{--- (3.1)}$$

$$- \left[ (N-n) \left( \frac{\lambda_0 - \epsilon}{n+1} \right) + [(\mu - \epsilon) + (n-1)\xi p] \right] P_n(0) + (N-n+1) \frac{(\lambda_0 - \epsilon)}{n} P_{n-1}(0) + [(\mu - \epsilon) + n\xi p] P_{n+1}(0) = 0; 1 \leq n \leq r-1 \quad \text{--- (3.2)}$$

$$- \left[ (N-r) \left( \frac{\lambda_0 - \epsilon}{r+1} \right) + [(\mu - \epsilon) + (r-1)\xi p] \right] P_r(0) + (N-r+1) \frac{(\lambda_0 - \epsilon)}{r} P_{r-1}(0) + [(\mu - \epsilon) + r\xi p] P_{r+1}(0) + [(\mu - \epsilon) + r\xi p] P_{r+1}(1) = 0 \quad \text{--- (3.3)}$$

$$- \left[ (N-n) \left( \frac{\lambda_0 - \epsilon}{n+1} \right) + [(\mu - \epsilon) + (n-1)\xi p] \right] P_n(0) + (N-n+1) \frac{(\lambda_0 - \epsilon)}{n} P_{n-1}(0) + [(\mu - \epsilon) + n\xi p] P_{n+1}(0) = 0; 1 \leq n \leq r \quad \text{--- (3.4)}$$

$$- \left[ (N-R+1) \frac{(\lambda_0 - \epsilon)}{R} + [(\mu - \epsilon) + (R-2)\xi p] \right] P_R(0) + (N-R+2) \frac{(\lambda_0 - \epsilon)}{R-1} P_{R-2}(0) = 0 \quad \text{--- (3.5)}$$

$$- \left[ (N-r-1) \left( \frac{\lambda_1 - \epsilon}{r+2} \right) + [(\mu - \epsilon) + r\xi p] \right] P_{r+1}(1) + [(\mu - \epsilon) + (r+1)\xi p] P_{r+2}(1) = 0 \quad \text{--- (3.6)}$$

$$- \left[ (N-n) \left( \frac{\lambda_1 - \epsilon}{n+1} \right) + [(\mu - \epsilon) + (n-1)\xi p] \right] P_n(1) + (N-n+1) \frac{(\lambda_1 - \epsilon)}{n} P_{n-1}(1) + \left[ \frac{(\lambda_0 - \epsilon)^{n-r-1} (N-r-1)}{n P_{n-r-1}} + \frac{(\lambda_0 - \epsilon)^{n-r-2}}{n P_{n-r-2}} (N-r-2)_{n-r-2} [(\mu - \epsilon) + r\xi p] \right] + \dots + [(\mu - \epsilon) + r\xi p] \dots [(\mu - \epsilon) + (n-2)\xi p] \quad r+1 \leq n \leq R-1 \quad \text{--- (3.12)}$$

using (3.12) in (3.5), we get

$$P_{r+1}(1) = \frac{[N - (R-1)](N)_{R-1} \frac{(\lambda_0 - \epsilon)^R}{R!} \prod_{l=0}^r \frac{1}{[(\mu - \epsilon) + l\xi p]} P_0(0)}{\frac{(\lambda_0 - \epsilon)^{R-r-1}}{R P_{R-r-1}} (N-r-1)_{R-r-1} + \frac{(\lambda_0 - \epsilon)^{R-r-2}}{R P_{R-r-2}} (N-r-2)_{R-r-2} [(\mu - \epsilon) + r\xi p] + \dots + \frac{(\lambda_0 - \epsilon)}{R} (N-R+1) [(\mu - \epsilon) + r\xi p] \dots [(\mu - \epsilon) + (R-3)\xi p]}$$

$$[(\mu - \epsilon) + n\xi p] P_{n+1}(1) = 0 \quad \text{--- (3.7)}$$

$$- \left[ (N-R) \left( \frac{\lambda_1 - \epsilon}{R+1} \right) + [(\mu - \epsilon) + (R-1)\xi p] \right] P_R(1) + (N-R+1) \frac{(\lambda_1 - \epsilon)}{R} P_{R-1}(1) + (N-R+1) \frac{(\lambda_0 - \epsilon)}{R} P_R(0) + [(\mu - \epsilon) + R\xi p] P_{R+1}(1) = 0 \quad \text{--- (3.8)}$$

$$- \left[ (N-n) \left( \frac{\lambda_1 - \epsilon}{n+1} \right) + [(\mu - \epsilon) + (n-1)\xi p] \right] P_n(1) + (N-n+1) \frac{(\lambda_1 - \epsilon)}{n} P_{n-1}(1) + [(\mu - \epsilon) + n\xi p] P_{n+1}(1) = 0 \quad \text{--- (3.9)}$$

$$- (\mu - \epsilon) P_N(1) + \frac{(\lambda_1 - \epsilon)}{N} P_{N-1}(1) = 0 \quad \text{--- (3.10)}$$

from (3.1) and (3.2), we get

$$P_n(0) = \frac{(N)_n (\lambda_0 - \epsilon)^n}{n! \prod_{l=0}^{n-1} [(\mu - \epsilon) + l\xi p]} P_0(0), n = 0, 1, 2, 3, \dots, r \quad \text{--- (3.11)}$$

where  $(N)_n = N(N-1)(N-2) \dots (N-n+1)$ .

using (3.11) in (3.3), we get

$$P_{r+1}(0) = \frac{(N)_{r+1} (\lambda_0 - \epsilon)^{r+1}}{(r+1)! \prod_{l=0}^r [(\mu - \epsilon) + l\xi p]} P_0(0) - P_{r+1}(1)$$

using the above result and (3.11) in (3.4), we get

$$P_n(0) = \frac{(N)_n (\lambda_0 - \epsilon)^n}{n! \prod_{l=0}^{n-1} [(\mu - \epsilon) + l\xi p]} P_0(0) - \frac{P_{r+1}(1)}{\prod_{l=r+1}^{n-1} [(\mu - \epsilon) + l\xi p]}$$

-----(3.13)

using (3.13), (3.6) in (3.7), we recursively derive

$$P_n(1) = \frac{P_{r+1}(1)}{\prod_{l=r+1}^{n-1} [(\mu - \epsilon) + l\xi p]} \left[ \frac{(\lambda_1 - \epsilon)^{n-r-1}}{nP_{n-r-1}} (N-r-1)_{n-r-1} + \frac{(\lambda_1 - \epsilon)^{n-r-2}}{nP_{n-r-2}} (N-r-2)_{n-r-2} \right] \left[ [(\mu - \epsilon) + r\xi p] + \dots + [(\mu - \epsilon) + (n-2)\xi p] \right]$$

-----(3.14)

where  $P_{r+1}(1)$  is given by (3.13)

using (3.12) and (3.14) in (3.8), we get

$$P_{R+1}(1) = \frac{P_{r+1}(1)}{\prod_{l=r+1}^R [(\mu - \epsilon) + l\xi p]} \left[ \frac{(\lambda_1 - \epsilon)^{R-r}}{(R+1)P_{R-r}} (N-r-1)_{R-r} + \frac{(\lambda_1 - \epsilon)}{(R+1)P_{R-r-1}} (N-r-2)_{R-r-1} [(\mu - \epsilon) + r\xi p] + \dots + \frac{(\lambda_1 - \epsilon)}{R+1} (N-R)[(\mu - \epsilon) + r\xi p] \dots [(\mu - \epsilon) + r\xi p] \right] \text{-----(3.15)}$$

using the above result in (3.9) and (3.10), we recursively derive

$$P_n(1) = \frac{P_{r+1}(1)}{\prod_{l=r+1}^{n-1} [(\mu - \epsilon) + l\xi p]}$$

$$\left[ \frac{(\lambda_1 - \epsilon)^{n-r-1}}{nP_{n-r-1}} (N-R-1)_{n-r-1} + \frac{(\lambda_1 - \epsilon)^{n-r-2}}{nP_{n-r-2}} (N-R-1)_{n-r-2} \right] \left[ [(\mu - \epsilon) + r\xi p] + \dots + \frac{(\lambda_1 - \epsilon)^{n-R}}{nP_{n-R}} (N-R-1)_{n-R-1} \right] \left[ [(\mu - \epsilon) + r\xi p][(\mu - \epsilon) + (r+1)\xi p] \dots [(\mu - \epsilon) + (R-2)\xi p] \right]$$

$R \leq n \leq K$

where  $P_{r+1}(1)$  is given by (3.13)

Thus from (3.11) to (3.15), we find that the steady state probabilities are expressed in terms of  $P_0(0)$ .

#### 4. Characteristics of the Model

The following system characteristics are considered and their analytical results are derived in this system.

- i) The probability  $P(0)$  that the system is in faster rate of arrivals.
- ii) The probability  $P(1)$  that the system is in slower rate of arrivals.
- iii) The probability  $P(0)$  that the system is empty.
- iv) The expected number of customers in the system  $L_{s_0}$ , when the system is in the faster rate of arrivals.
- v) The expected number of customers in the system  $L_{s_1}$ , when the system is in the slower rate of arrivals.
- vi) The expected waiting time of the customer in the system  $W_s$

The probability that the system is in faster rate of arrivals is

$$P(0) = \sum_{n=0}^k P_n(0)$$

$$P(0) = \sum_{n=R}^K P_n(0) + \sum_{n=r+1}^{R-1} P_n(0) + \sum_{n=R}^K P_n(0)$$

$$= \sum_{n=0}^{R-1} \left[ \frac{1}{n!} \frac{(\lambda_0 - \epsilon)^n}{\prod_{l=0}^{n-1} [(\mu - \epsilon) + l\xi p]} \right] + \sum_{n=r+1}^{R-1} P_n(0) + \sum_{n=R}^k P_n(0)$$

$$\sum_{n=R}^k P_n(0)$$

Since  $P_n(0)$  exists only when  $n=0, 1, 2, \dots, r-1, r, r+1, r+2, \dots, R-2, R-1$ .

We get

$$P(0) = \sum_{n=0}^r P_n(0) + \sum_{n=r+1}^{R-1} P_n(0) \text{-----(4.1)}$$

From (3.11), (3.12), (3.13) and (4.1), we get

$$P(0) = \frac{1}{n!} \frac{(\lambda_0 - \epsilon)^n}{\prod_{l=0}^{n-1} [(\mu - \epsilon) + l\xi p]} (N)_n P_0(0) - \sum_{n=r+1}^{R-1} \frac{A}{B} \frac{(\lambda_0 - \epsilon)^R}{R!} \frac{1}{\prod_{l=0}^{n-1} [(\mu - \epsilon) + l\xi p]} (N)_R P_0(0) \text{----}$$

(4.2)  
where

$$A = \frac{(\lambda_0 - \epsilon)^{n-r-1}}{nP_{n-r-1}} [(N-r-1)_{n-r-1} + \frac{(\lambda_0 - \epsilon)^{n-r-2}}{nP_{n-r-2}} [N-r-2]_{n-r-2} \\ + \dots + [(\mu - \epsilon) + r\xi p] + \dots + [(\mu - \epsilon) + r\xi p][(\mu - \epsilon) + (r+1)\xi p] \dots [(\mu - \epsilon) + (n-2)\xi p]$$

$$B = \frac{(\lambda_0 - \epsilon)^{R-r-1}}{RP_{R-r-1}} (N-r-1)_{R-r-1} + \frac{(\lambda_0 - \epsilon)^{R-r-2}}{RP_{R-r-2}} [N-r-2]_{R-r-2} [(\mu - \epsilon) + r\xi p] \\ + \dots + [(\mu - \epsilon) + r\xi p][(\mu - \epsilon) + (r+1)\xi p] \dots [(\mu - \epsilon) + (R-2)\xi p]$$

The probability that the system is in slower rate of arrivals is

$$P(1) = \sum_{n=0}^K P_n(1) \qquad P(1) = \sum_{n=r+1}^R \left[ \frac{C (\lambda_0 - \epsilon)^R}{B R!} \frac{1}{\prod_{l=0}^{n-1} [(\mu - \epsilon) + l\xi p]} (N)_R P_0(0) \right] + \\ = \sum_{n=0}^K P_n(1) + \sum_{n=r+1}^{R-1} P_n(1) + \sum_{n=R}^K P_n(1)$$

Since  $P_n(1)$  exists only when  $n = r+1, r+2, \dots, R-2, R-1, \dots, k$

$$\text{We get } P(1) = \sum_{n=r+1}^R P_n(1) + \sum_{n=R+1}^K P_n(1) \quad \text{-----(4.3)}$$

$$\sum_{n=R+1}^K \left[ \frac{D (\lambda_0 - \epsilon)^R}{B R!} \frac{1}{\prod_{l=0}^{n-1} [(\mu - \epsilon) + l\xi p]} (N)_R P_0(0) \right] \quad \text{-----(4.4)}$$

from (3.14), (3.15) and (4.3), we get

where

$$C = \frac{(\lambda_1 - \epsilon)^{n-r-1}}{nP_{n-r-1}} (N-r-1)_{n-r-1} + \frac{(\lambda_1 - \epsilon)^{n-r-2}}{nP_{n-r-2}} (N-r-2)_{n-r-2} [(\mu - \epsilon) + r\xi p] \\ + \dots + [(\mu - \epsilon) + r\xi p][(\mu - \epsilon) + (r+1)\xi p] \dots [(\mu - \epsilon) + (n-2)\xi p]$$

$$D = \frac{(\lambda_1 - \epsilon)^{K-1-r}}{KP_{K-r-1}} (N-R-1)_{n-R-1} + \frac{(\lambda_1 - \epsilon)^{K-r-2}}{KP_{K-r-2}} (N-R-1)_{n-R-2} [(\mu - \epsilon) + r\xi p] \\ + \dots + \frac{(\lambda_1 - \epsilon)^{K-R}}{KP_{K-R}} (N-R-1)_{n-R} [(\mu - \epsilon) + r\xi p] \dots [(\mu - \epsilon) + (R-2)\xi p]$$

The probability  $P_0(0)$  that the system is empty can be calculate from the normalizing condition.

$$P(0) + P(1) = 1$$

$$P_0(0) = \frac{1}{1 + \sum_{n=1}^{R-1} \left[ \frac{1}{n!} \frac{(\lambda_0 - \epsilon)^n (N)_n}{\prod_{l=0}^{n-1} [(\mu - \epsilon) + l\xi p]} \right] - \sum_{n=r+1}^{R-1} \left[ \frac{A (\lambda_0 - \epsilon)^R}{B R!} \frac{(N)_R}{\prod_{l=0}^{n-1} [(\mu - \epsilon) + l\xi p]} \right]} + \sum_{n=r+1}^R \left[ \frac{C (\lambda_0 - \epsilon)^R}{B R!} \frac{1}{\prod_{l=0}^{n-1} [(\mu - \epsilon) + l\xi p]} (N)_R \right] + \sum_{n=R+1}^K \left[ \frac{D (\lambda_0 - \epsilon)^R}{B R!} \frac{1}{\prod_{l=0}^{n-1} [(\mu - \epsilon) + l\xi p]} (N)_R \right]$$

where A, B, C and D is given by (4.2) and (4.4)  
 Now, we calculate the expected number of customers in the system. Let  $L_s$  denote the average number of customers in the system, then we have

$$L_s = L_{s_0} + L_{s_1}$$

where

$$L_{s_0} = \sum_{n=0}^r nP_n(0) + \sum_{n=r+1}^{R-1} nP_n(0)$$

$$\text{and } L_{s_1} = \sum_{n=r+1}^{R-1} nP_n(1) + \sum_{n=R}^K nP_n(1)$$

using Little's formula, the expected waiting time of the customer in the system is calculated as  $W_s = \frac{L_s}{\lambda}$  where

$$\bar{\lambda} = \lambda_0 P(0) + \lambda_1 P(1)$$

### 5. Numerical Illustration

r	R	K	N	$\lambda_0$	$\lambda_1$	$\mu$	$\epsilon$	$\xi$	p	$P_0(0)$	P(0)	P(1)	$L_s$	$W_s$
4	6	10	10	4	3	5	1	1	1	0.00899	0.75108	0.25991	3.44242	0.92436
4	6	10	10	4	3	5	0	1	1	0.00539	0.62451	0.38009	3.85820	1.06924
4	6	10	10	4	3	5	0.5	1	1	0.00687	0.68645	0.32667	3.66175	0.99888
4	6	10	10	2	1	5	0.5	1	1	0.08143	0.89715	0.02140	2.01522	1.10987
4	6	10	10	4	3	6	0.5	1	1	0.01213	0.73378	0.27408	3.42411	0.92608
4	6	10	10	4	3	8	0.5	1	1	0.02807	0.83031	0.17161	2.98094	0.80217
4	6	10	10	4	4	5	0.5	1	1	0.00661	0.66096	0.34242	3.74837	0.94333
4	6	10	10	4	4	5	0	1	1	0.00514	0.59629	0.41856	3.95982	0.99507
4	6	10	10	3	2	5	0.5	1	1	0.02250	0.84850	0.16899	2.91491	1.03974
4	6	10	10	5	5	5	0	1	1	0.00179	0.44349	0.56470	4.49610	0.90083
4	6	10	10	6	3	5	0.5	1	1	0.00096	0.44458	0.56445	4.41170	1.02577
4	6	10	10	6	5	4	0.5	1	1	0.00036	0.34146	0.66817	4.76348	0.89377
4	6	10	10	4	3	5	1	0	0	0.00899	0.75108	0.25991	3.44242	0.92436
4	6	10	10	4	3	5	0.5	0.5	0.5	0.00300	0.53068	0.47630	4.21204	1.19944
4	6	10	10	2	1	5	0	1	1	0.05232	0.89522	0.15245	2.35899	1.28004

### 6. Conclusion

It is observed from the numerical value that when the mean service rate increases and the other parameters are kept fixed,  $P_0(0)$ ,  $P(0)$  increases and  $P(1)$  decrease  $L_s$  decrease and  $W_s$  decreases. When the arrival rate decreases (the other parameters are kept fixed),  $P_0(0)$ ,  $P(0)$  increases and  $P(1)$  decrease  $L_s$  increases and  $W_s$  decreases. When the mean dependence rate increases and the other parameters are kept fixed,  $P_0(0)$  and  $P(0)$  increase,  $P(1)$  decrease. When the value of  $\xi$  increases and the other parameters are kept fixed  $P_0(0)$  and  $P(0)$  increase,  $P(1)$  decrease.

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