Identifying Irrationals

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Abstract: ***Irrational numbers have always been a fascination to mathematiciansfor several millennia. This is because Irrational numbers neither terminate nor repeat in their decimal expansion. Hence exploring the next set of digits after a given decimal place has kept many mathematicians and computer scientists busy in past few decades. A classic example is exploration of digits of the most famous and important real number π. In this paper, I shall present a novel method with proof using analysis to find the rational numbers which are very good approximations to the given Irrational Number and present a more general method of finding approximations to all Algebraic numbers.***

Keywords: Irrational Numbers, Sequence and Sub-sequence, Cauchy Sequence, Convergence, Algebraic Numbers, Approximations

1. Introduction

Let us consider p to be a prime number. We emphasize the fact that prime numbers are natural numbers which has exactly two divisors. So 1 is excluded from this list. The set of primes numbers are 2,3,5,7,11,13, . . .

The study of behaviour of prime numbers has been source of investigation for many centuries and till today most intelligent minds in history of mathematics have spent considerable time in finding the patterns of these mysterious numbers.

Euclid in Third Century B.C. in his Ninth volume of "The Elements" has proved that there are "Infinitely many primes". The proof provided by Euclid is considered by many as one of the most beautiful in proving theorems.

The set of all real numbers R is the union of rational numbers and irrational numbers. The ratio of two co-prime integers where the denominator is non-zero is called a rational number. Hence a rational number can be represented as $\frac{p}{q}$ where $(p,q)=1, q\neq 0$.

The real numbers which are not represented in above form are called irrational numbers. Thus, it is impossible to express any irrational number as quotient of two integers. So, with respect to real number system, the irrationals are precisely the complements of rational numbers.

As a consequence of Bolzano-Weierstrass theorem, it is known that between any two rational numbers there are infinitely many irrational numbers and conversely between any two irrational numbers there are infinitely many rational numbers. With this idea, we try to explore the irrational numbers of the form $\sqrt{p}$, when p is prime through its neighbouring rational numbers. I introduce a novel method for doing so and prove that this method always yield irrational numbers of the form $\sqrt{p}$

2. The Square Root of Two

The square root of two denoted by $\sqrt{2}$ has a special importance in history of mathematics, since it is the first irrational number to be discovered. It was believed that one of the followers of Pythagoras, discovered that the length of a diagonal of a unit side square is exactly $\sqrt{1^2+1^2}=\sqrt{2}$. This discovery shattered the belief of Pythagoreans who till then claimed that everything in the universe can be expressible as ratio of two integers which are rational numbers. Legend has it that the discoverer of this idea was thrown in the sea.

After this incident, mathematicians like Theodorus proved numbers of the form $\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{7},\sqrt{11},\sqrt{13},\sqrt{17}$ are irrationals and stopped at $\sqrt{17}$. Today there exist innumerable ways of proving the irrationality of these numbers.

3. Method for Computing $\sqrt{2}$:

We now provide an elegant method to compute $\sqrt{2}$ .The general idea for computing $\sqrt{2}$ is to iterate with the help of a function repeatedly, until we get desired accuracy to approximate $\sqrt{2}$. The following function is used for this purpose.

$$f\left(\frac{a}{b}\right)=\frac{a+2b}{a+b}$$

We now begin our iteration with $a=1, b=1$ and continue repeatedly until we get desired rational number closer to $\sqrt{2}$.

$$f\left(\frac{1}{1}\right)=\frac{1+(2\times 1)}{1+1}=\frac{3}{2},\ f\left(\frac{3}{2}\right)=\frac{3+(2\times 2)}{3+2}=\frac{7}{5},\ f\left(\frac{7}{5}\right)=\frac{7+(2\times 5)}{7+5}=\frac{17}{12},\ f\left(\frac{17}{12}\right)=\frac{17+(2\times 12)}{17+12}=\frac{41}{29}$$

$$f\left(\frac{41}{29}\right)=\frac{41+(2\times 29)}{41+29}=\frac{99}{70},\ f\left(\frac{99}{70}\right)=\frac{99+(2\times 70)}{99+70}=\frac{239}{169},\ldots$$

From the six iterate values that are provided above, it is easy to see that the numbers $\frac{3}{2},\frac{17}{12},\frac{99}{70},\ldots$ are more than $\sqrt{2}$ and the numbers $\frac{7}{5},\frac{41}{29},\frac{239}{169},\ldots$ are less than $\sqrt{2}$. Now these numbers on either side of $\sqrt{2}$ will in fact can be shown to converge to $\sqrt{2}$. We see this assertion in the form of a theorem.

3.1 Theorem 1:

The function $f\left(\frac{a}{b}\right)=\frac{a+2b}{a+b}$ always produces iterates which converge to $\sqrt{2}$.

Proof: Let $x_n=\frac{a}{b}, x_{n+1}=\frac{a+2b}{a+b}$. Then we can express the function *f* in the form $f(x_n)=x_{n+1}$. Now considering x_{n+1} and relating it with x_n we get

$$x_{n+1}=\frac{a+2b}{a+b}=\frac{\frac{a}{b}+2}{\frac{a}{b}+1}=\frac{x_n+2}{x_n+1}$$

Now considering the difference of x_{n+1} and x_n, we get $x_{n+1}-x_n=\frac{2-x_n^{\ 2}}{x_n+1}=\frac{2-x_n^{\ 2}}{x_n+x_1}$, since $x_1=1$. Continuing in this fashion and considering the difference of subsequent terms from x_n we get the following equations:

$$x_{n+2}-x_n=\frac{2-x_n^{\ 2}}{x_n+x_2},\ x_{n+3}-x_n=\frac{2-x_n^{\ 2}}{x_n+x_3},\ x_{n+4}-x_n=\frac{2-x_n^{\ 2}}{x_n+x_4}$$

In general, for any natural number *k*, we have

$$x_{n+k}-x_n=\frac{2-x_n^{\ 2}}{x_n+x_k}$$

Since $x_n, x_k\geq 1$, we have $\frac{1}{x_n+x_k}\leq\frac{1}{2}$ and so

$$x_{n+k}-x_n\leq\frac{\left|2-x_n^{\ 2}\right|}{2}\leq\frac{1}{2}$$

In fact, for large values of *k*. we get $\left|x_{n+k}-x_n\right|<\in$. Hence $\left\{x_n\right\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers. Since any Cauchy sequence of real numbers is convergent, it follows that the sequence $\left\{x_n\right\}_{n=1}^{\infty}$ is convergent.

Let $\lim_{n\to\infty}x_n=L$. Since $\left\{x_{n+1}\right\}_{n=1}^{\infty}$ is a subsequence of $\left\{x_n\right\}_{n=1}^{\infty}$, we should get $\lim_{n\to\infty}x_{n+1}=L$

Thus from the equation $x_{n+1}=\frac{x_n+2}{x_n+1}$ upon taking the limits on both sides we get $L=\frac{L+2}{L+1}$ from which $L=\sqrt{2}$.

So the function $f\left(\frac{a}{b}\right)=\frac{a+2b}{a+b}$ always produces numbers converging to $\sqrt{2}$ and this completes the proof.

4. Other Square Roots

Similar to **Theorem 1**, discussed above, we can prove the following theorem.

4.1 Theorem 2

The function $f\left(\frac{a}{b}\right)=\frac{a+nb}{a+b}$ always produces iterates converging to the irrational number $\sqrt{n}$, where $n\neq k^2$.

Proof: Let $x_n=\frac{a}{b}, x_{n+1}=\frac{a+nb}{a+b}$. Then as discussed with the case $\sqrt{2}$, we can show that

$$x_{n+k}-x_n\leq\frac{\left|n-x_n^{\ 2}\right|}{2}\leq\frac{1}{2}$$

Hence for large values of k, we have $\left|x_{n+k} - x_n\right| < \in$. Hence $\left\{x_n\right\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers which is convergent. Let $\lim_{n\to\infty} x_n = L$. Since $\left\{x_{n+1}\right\}_{n=1}^{\infty}$ is a subsequence of $\left\{x_n\right\}_{n=1}^{\infty}$, we should get $\lim_{n\to\infty} x_{n+1} = L$. Hence from $x_{n+1} = \dfrac{x_n + n}{x_n + 1}$ upon taking the limits on either sides, we get $L = \dfrac{L+n}{L+1}$. This gives $L = \sqrt{n}$. This proves our claim.

5. General Case

We now provide sequence of iterates converging to $\sqrt[n]{p} = (p)^{\frac{1}{n}}$ where $p \neq k^n$. For this, we consider the following theorem.

5.1 Theorem 3:

The function $f\left(\dfrac{a}{b}\right) = \dfrac{b^{n-2}(a+pb)}{a^{n-1}+b^{n-1}}$ always produces iterates which converge to $(p)^{\frac{1}{n}}$.

Proof: Assuming $x_n = \dfrac{a}{b}, x_{n+1} = \dfrac{x_n + p}{x_n^{\,n-1}+1}$ we find that $x_{n+1} - x_n = \dfrac{p - x_n^{\,n}}{x_n + 1} = \dfrac{p - x_n^{\,n}}{x_n + x_1}$ since $x_1 = 1$. In general, for any natural number k, we have

$$x_{n+k} - x_n \leq \frac{\left|p - x_n^{\,n}\right|}{2} \leq \frac{1}{2}$$

Hence for large values of k, we have $\left|x_{n+k} - x_n\right| < \in$. Hence $\left\{x_n\right\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers which is convergent. Let $\lim_{n\to\infty} x_n = L$. Hence from $x_{n+1} = \dfrac{x_n + p}{x_n^{\,n-1}+1}$, taking limits on either sides we get $L = \dfrac{L+p}{L^{n-1}+1}$. This simplifies to $L^n = p$ and so $L = (p)^{\frac{1}{n}}$.

Thus the function $f\left(\dfrac{a}{b}\right) = \dfrac{b^{n-2}(a+pb)}{a^{n-1}+b^{n-1}}$ always produces iterates which converges to $(p)^{\frac{1}{n}}$ as required.

6. Conclusion

From the three theorems discussed above, we find an elegant and novel way of approximating all Algebraic numbers (numbers which are roots of polynomial equations with rational coefficients) efficiently. The fact that these schemes provide easy way of finding the approximations, it would be very useful in many Science and Engineering applications. From mathematical perspective, this method would have made Pythagoreans and Theodorus much happy.

References

[1] James R. Choike (1980). "The Pentagram and the Discovery of an Irrational Number". The Two-Year College Mathematics Journal.
[2] Robert L. McCabe (1976). "Theodorus' Irrationality Proofs". Mathematics Magazine.
[3] Julian Havil (2014). “The Irrationals: A story of the numbers you can’t count on”, Princeton University Press.
[4] T. K. Puttaswamy (2012), "The Accomplishments of Ancient Indian Mathematicians", Elsevier