

Chain Conditions on Fuzzy G-Modules

Souriar Sebastian¹, Prathish Abraham¹

^{1,2}Department of Mathematics, St. Albert's College, Ernakulam, Kochi-682018, Kerala (India)

Abstract: The concept of fuzzy G-Modules and its properties including complete reducibility, semi simplicity etc. are already defined. In this paper, the concept of minimal and maximal sub modules is defined on fuzzy G-modules. The chain conditions for fuzzy G-modules are also defined and some properties of both ascending and descending chains of sub modules are established.

Keywords: Fuzzy-G-Modules, Proper fuzzy G sub modules, minimal and maximal sub modules of a fuzzy G module, chain conditions for fuzzy G modules

1. Introduction

The introduction of fuzzy sets by Zadeh led way to the fuzzification of Algebraic structures. Fuzzy groups and groupoids are defined by Rosenfield^[9]. The concept of fuzzy G modules was introduced by the Sherry Fernandez^[10]. Semi simplicity of fuzzy G modules and its relations with complete reducibility and injectivity were studied by the author^[8].

In this paper the concept of minimal and maximal sub modules is defined on fuzzy G-modules. The descending and ascending chain conditions for fuzzy G-modules are defined and some properties of these chains of fuzzy G sub modules are established.

2. Preliminaries

A vector space M over a field K is said to be a **G-Module** if for every $g \in G$ and $m \in M$ there exists a product 'gm' called action of G on M satisfying

- (i) $1_G m = m, \forall m \in M$
- (ii) $(gh)m = g(hm), \forall g, h \in G, m \in M$
- (iii) $g(k_1 m_1 + k_2 m_2) = k_1(gm_1) + k_2(gm_2), \forall g \in G, m_1, m_2 \in M, k_1, k_2 \in K$

As an example, for $G = \{1, -1, i, -i\}$, $M = C^n$ over C is a G-Module under usual addition and multiplication.

A subspace of M, which itself is a G-Module with the same action is called **G sub module**. A non-zero G Module M is **irreducible** if the only G sub modules of M are M and $\{0\}$. otherwise it is **reducible**. A non-zero G module M is **completely reducible** if for every G sub module N of M there exists a G sub module N^* of M such that $M = N \oplus N^*$. A G Module M is **semi simple** if there exists a family of irreducible G sub modules M_i such that $M = \bigoplus_{i=1}^n M_i$

A fuzzy G module over a G Module M is a fuzzy set μ on M such that

- (i) $\mu(ax + by) \geq \min(\mu(x), \mu(y)), \forall a, b \in K$ and $x, y \in M$
- (ii) $\mu(gm) \geq \mu(m), \forall m \in M$ and $g \in G$.

For example, For $G = \{1, -1\}$ and $M = Q\sqrt{2}$ over Q , The fuzzy set μ on M defined as $\mu(a + b\sqrt{2}) = 1$ if $a = 0, b = 0$

- = .8 if $a \neq 0, b = 0$
- = .2 if $b \neq 0$ is a fuzzy G module over M.

The standard fuzzy intersection of finite no of fuzzy G Modules is again a fuzzy G module. The collection of all elements in the universal set with membership value greater than a given α , $\alpha \in [0,1]$ is called an **α cut** of the fuzzy G Module μ , denoted by $\mu^{\alpha+}$. The 0 cut, μ^{0+} consisting of all elements with a non zero membership is called the **support of μ** , denoted by $Supp \mu$.

3. Minimal and Maximal Fuzzy G sub modules

3.1 Definition

A fuzzy G sub module ϑ of a fuzzy G module μ on a G module M is called a minimal fuzzy G sub module if $\vartheta \neq \chi_{\{0\}}$ and $Supp \vartheta \subseteq Supp \theta$ for all fuzzy G sub modules θ of μ .

3.2 Definition

A fuzzy G sub module ϑ of a fuzzy G module μ on a G module M is called a maximal fuzzy G sub module if $\vartheta \neq \mu$ and $Supp \vartheta \supseteq Supp \theta$ for all fuzzy G sub modules θ of μ .

4. Descending and Ascending Chain Conditions on Fuzzy G sub modules

4.1 Definition

A fuzzy G module μ of a G Module M is said to satisfy the **descending chain condition (d.c.c)** if for any descending chain $\{\mu_i\}$ of fuzzy G sub modules of μ , the corresponding chain of supports, $\{Supp \mu_i\}$ satisfies the descending chain condition for G modules. There exists, an integer r satisfying $Supp \mu_r = Supp \mu_{r+1} = Supp \mu_{r+2} = \dots$

A fuzzy G module μ of a G Module M is said to satisfy the **ascending chain condition (a.c.c)** if for any ascending chain $\{\mu_i\}$ of fuzzy G sub modules of μ , the corresponding chain of supports $\{Supp \mu_i\}$ satisfies the ascending chain condition for G modules. Hence $Supp \mu_r = Supp \mu_{r+1} = Supp \mu_{r+2} = \dots$ for some integer r

4.2 Proposition

The following statements are equivalent for a fuzzy G module ϑ of a G Module M

- 1) The descending chain condition holds for ϑ
- 2) Any non empty family of fuzzy G sub modules of ϑ has a minimal sub module

Proof: **1) \Rightarrow 2)** Assume that descending chain condition holds for ϑ . Consider a chain $\{\mu_i\}$ of fuzzy G sub modules of ϑ . Take any fuzzy G sub module μ_i form the chain. If μ_i is the minimal then the proof is complete. Otherwise, there is fuzzy G sub module μ_j with $Supp\mu_j \subseteq Supp\mu_i$. If μ_j is the minimal one, the proof is complete. If not, we continue the process and if we cannot find a minimal element in finite steps, we get chain of fuzzy G sub modules whose corresponding chain of supports is non stationary. This contradicts our assumption and proves the existence of a minimal fuzzy G sub module for μ .

1) \Rightarrow 2) Assume that, every non empty family of fuzzy G sub modules of μ has a minimal element. Hence there is a fuzzy G sub module

μ_i , such that for any fuzzy G sub module μ_j in the family, $Supp\mu_i \subseteq Supp\mu_j$. Consider any chain of fuzzy G sub modules of μ . Then it must have a minimal element μ_r satisfying $Supp\mu_r \subseteq Supp\mu_j$ for all $j \geq r$. But as the chain is a descending chain we can conclude that $Supp\mu_r = Supp\mu_j$ for all $j \geq r$. Hence μ satisfies the descending chain condition.

4.3 Proposition

If a fuzzy G module μ on a G module M satisfies descending chain condition then all its fuzzy G sub modules and quotient fuzzy G sub modules satisfy the d.c.c.

Proof:

Assume ϑ is a fuzzy G sub module of μ . Then any descending chain $\{\vartheta_i\}$ of fuzzy G sub modules of ϑ is a descending chain of fuzzy G sub modules of μ . Hence $\{supp\vartheta_i\}$ satisfies the d.c.c and thereby proves that the fuzzy G sub module ϑ of μ satisfies d.c.c.

Let $\mu_{(N)}^M$ is the quotient fuzzy G sub module of μ on $\frac{M}{N}$. Then any descending chain $\{\mu_i(\frac{M}{N})\}$ of fuzzy G sub modules of $\mu_{(N)}^M$ corresponds to a descending chain of fuzzy G sub modules $\{\mu_i\}$ of μ with $supp\mu_i(\frac{M}{N}) = supp\mu_i$. As μ satisfies d.c.c, $\{supp\mu_i\}$ satisfies d.c.c of G modules. This proves that $\mu_{(N)}^M$ satisfies d.c.c.

4.4 Proposition

Let μ is a fuzzy G module on a G module M and ϑ is a fuzzy G sub module of μ defined on a G sub module N of M. If ϑ and $\mu_{(N)}^M$ satisfies d.c.c then μ also satisfies d.c.c.

Proof: Consider a descending chain $\{\mu_i\}$ of fuzzy G sub modules of μ . Then $\{\mu_i \cap \vartheta\}$ is descending chain of fuzzy G sub modules of ϑ . As ϑ satisfies d.c.c, there is an integer r such that $supp(\mu_r \cap \vartheta) = supp(\mu_{r+1} \cap \vartheta) = supp(\mu_{r+2} \cap \vartheta) = \dots$

Also $\{\mu_i + \vartheta\}_{\{supp(\frac{\mu_i + \vartheta}{N})\}}$ is descending chain of fuzzy G sub modules of $\mu_{(N)}^M$. As $\mu_{(N)}^M$ satisfies d.c.c there is an integer s such that $\frac{supp(\mu_s + \vartheta)}{N} = \frac{supp(\mu_{s+1} + \vartheta)}{N} = \frac{supp(\mu_{s+2} + \vartheta)}{N} = \dots$

Hence we have

1. $\mu_n \geq \mu_{n+1}$ for all n,
2. $supp(\mu_n \cap \vartheta) = supp(\mu_{n+1} \cap \vartheta)$ for all $n \geq r$,
3. $\frac{supp(\mu_n + \vartheta)}{N} = \frac{supp(\mu_{n+1} + \vartheta)}{N}$ for all $n \geq s$,
4. $supp(\mu_n + \vartheta) \cong \frac{supp\mu_n}{Supp\{\mu_n \cap \vartheta\}}$

Hence $\frac{supp\mu_n}{Supp\{\mu_n \cap \vartheta\}} = \frac{supp(\mu_n + \vartheta)}{N} = \frac{supp(\mu_{n+1} + \vartheta)}{N} = \frac{supp\mu_{n+1}}{Supp\{\mu_{n+1} \cap \vartheta\}}$

and by using 2 we prove that $supp(\mu_n) = supp(\mu_{n+1})$ for all $n > \max(r, s)$. Hence satisfies d.c.c.

4.5 Proposition

The direct sum of finitely many fuzzy G sub modules on a G module M satisfying d.c.c satisfies d.c.c

Proof: Let $\mu_1, \mu_2, \dots, \mu_n$ be a finite family of fuzzy G sub modules on a G module M satisfying d.c.c. We use Mathematical Induction to show that the direct sum $\mu = \bigoplus_{i=1}^n \mu_i$ of these n fuzzy G sub modules also satisfy d.c.c. Let $N = \bigoplus_{i=1}^n supp\mu_i$. For $n = 1$, the proof is trivial. For $n \geq 2$ we assume that $\bigoplus_{i=1}^{n-1} \mu_i$ satisfies d.c.c. Then, $N' = \bigoplus_{i=1}^{n-1} supp\mu_i$ satisfies d.c.c.

We have, $\frac{N}{supp\mu_n} = \frac{N' + supp\mu_n}{supp\mu_n} = \frac{N'}{N' \cap supp\mu_n} + \frac{supp\mu_n}{supp\mu_n}$.

As N' satisfy d.c.c, $\frac{N'}{N' \cap supp\mu_n}$ satisfies d.c.c. Hence $\frac{N}{supp\mu_n}$

satisfies d.c.c. Since $supp\mu_n$ and $\frac{N}{supp\mu_n}$ satisfies d.c.c N also satisfies d.c.c. Since $supp\bigoplus_{i=1}^n \mu_i = \bigoplus_{i=1}^n supp\mu_i = N$, the proof is complete by mathematical induction.

4.6 Proposition

The following statements are equivalent for a fuzzy G module ϑ of a G Module M

- 1) The ascending chain condition holds for ϑ
- 2) Any non empty family of fuzzy G sub modules of ϑ has a maximal sub module

Proof: **1) \Rightarrow 2)** Assume that ascending chain condition holds for μ . Consider a family $\{\mu_i\}$ of fuzzy G sub modules of μ . Take any fuzzy G sub module μ_i form the family. If μ_i is the maximal then the proof is complete. Otherwise there is fuzzy G sub module μ_j with $Supp\mu_j \supseteq Supp\mu_i$. If μ_j is the maximal, the proof is complete. If not, we continue the process and if we cannot find a maximal element in finite steps, we get an ascending chain of fuzzy G sub modules whose corresponding chain of supports is non stationary.

This contradicts our assumption and proves the existence of a maximal fuzzy G sub module for μ .

1) \Rightarrow 2) Assume that every non empty family of fuzzy G sub modules of μ has a maximal element. Hence there is a fuzzy G sub module μ_i such that, for any fuzzy G sub modules μ_j in the family $Supp\mu_i \supseteq Supp\mu_j$. Consider a chain of fuzzy G sub modules of μ and they must have a maximal element μ_r satisfying $Supp\mu_r \supseteq Supp\mu_j$ for all $j \geq r$. But as the chain is a ascending chain we can conclude that $Supp\mu_r = Supp\mu_j$ for all $j \geq r$. Hence μ satisfies the ascending chain condition.

4.7 Proposition

If a fuzzy G sub module μ on a G module M satisfies ascending chain condition then all its fuzzy G sub modules and quotient fuzzy G sub modules satisfy the a.c.c.

Proof:

Assume ϑ is a fuzzy G sub module of μ . Then any ascending chain $\{\vartheta_i\}$ of fuzzy G sub modules of ϑ is a ascending chain of fuzzy G sub modules of μ . Hence $\{supp\vartheta_i\}$ satisfies the a.c.c and thereby proves that the fuzzy G sub module ϑ of μ satisfies a.c.c.

Let $\mu_{\frac{M}{N}}$ is the quotient fuzzy G sub module of μ on M/N .

Then any ascending chain $\{\mu_{i\frac{M}{N}}\}$ of fuzzy G sub modules of $\mu_{\frac{M}{N}}$ corresponds to an ascending chain of fuzzy G sub modules $\{\mu_i\}$ of μ with $supp\mu_{i\frac{M}{N}} = supp\mu_i$. As μ satisfies a.c.c, $\{supp\mu_i\}$ satisfies a.c.c of G modules. This proves that $\mu_{\frac{M}{N}}$ satisfies a.c.c.

4.8 Proposition

Let μ is a fuzzy G module on a G module M and ϑ is a fuzzy G sub module of μ defined on a G sub module N of M. If ϑ and $\mu_{\frac{M}{N}}$ satisfies a.c.c., then μ also satisfies a.c.c.

Proof: Consider an ascending chain $\{\mu_i\}$ of fuzzy G sub modules of μ . Then $\{\mu_i \cap \vartheta\}$ is ascending chain of fuzzy G sub modules of ϑ . As ϑ satisfies a.c.c there is an integer r such that $supp(\mu_r \cap \vartheta) = supp(\mu_{r+1} \cap \vartheta) = supp(\mu_{r+2} \cap \vartheta) = \dots$

Also $\{\mu_i + \vartheta\}_{\{supp\frac{(\mu_i + \vartheta)}{N}\}}$ is an ascending chain of fuzzy G sub modules of $\mu_{\frac{M}{N}}$. As $\mu_{\frac{M}{N}}$ satisfies a.c.c there is an integer

s such that $\frac{supp(\mu_s + \vartheta)}{N} = \frac{supp(\mu_{s+1} + \vartheta)}{N} = \frac{supp(\mu_{s+2} + \vartheta)}{N} = \dots$

Hence we have

1. $\mu_n \leq \mu_{n+1}$ for all n,
2. $supp(\mu_n \cap \vartheta) = supp(\mu_{n+1} \cap \vartheta)$ for all $n \geq r$,
3. $\frac{supp(\mu_n + \vartheta)}{N} = \frac{supp(\mu_{n+1} + \vartheta)}{N}$ for all $n \geq s$,
4. $supp(\mu_n + \vartheta) \cong \frac{supp\mu_n}{Supp\{\mu_n \cap \vartheta\}}$

Hence $\frac{supp\mu_n}{Supp\{\mu_n \cap \vartheta\}} = \frac{supp(\mu_n + \vartheta)}{N} = \frac{supp(\mu_{n+1} + \vartheta)}{N} = \frac{supp\mu_{n+1}}{Supp\{\mu_{n+1} \cap \vartheta\}}$ and by using 2 we prove that $supp(\mu_n) = supp(\mu_{n+1})$ for all $n > \max(r, s)$. Hence satisfies a.c.c.

References

- [1] Charles W Curtis, Irving Reiner; Representation Theory of Finite Groups and Associative Algebras, Wiley Eastern, 1962
- [2] Claus Michel Ringel and Jan Schroer: Representation Theory of Algebras I: Modules, 2007
- [3] George J Klir and Bo Yuan; Fuzzy sets and Fuzzy Logic: Theory and Applications, Prentice Hall, India, 1995
- [4] Hiram Paley and Paul M. Weichsel; *A First Course in Abstract Algebra*, Holt, Rinehart, Winston Inc. 1996
- [5] Lambek, Joachim, *Lectures on Rings and Modules*, Blaisdell Publishing Company, 1966
- [6] Musli C: *Representation of Finite Groups*, Hindustan Book Agency, India, 1993
- [7] Musli C; *Introduction to Rings and Modules*, Narosa Publishing House, India, 1992
- [8] Prathish Abraham, Souriar Sebastian: *Semi simple Fuzzy G- Modules*, Journal of Computer and Mathematics Sciences, Vol.3 (4), 458-463 (2012)
- [9] Rosenfield A; *Fuzzy Groups*, J Math Anal. Appl., 1971
- [10] Sherry Fernandez: *A Study of Fuzzy G- Modules*, PhD Thesis, MG University, Kerala, 2004