1, Semiglobal Cototal Domination on Graphs

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Abstract: ***A subset D of vertices of a connected graph G is called a semiglobal cototal dominating set if D is a dominating set for G and*** G^{sc} ***and <V-D> has no isolated vertices in G, where*** G^{sc} ***is the semi complementary graph of G. The semiglobal cototal domination number is the minimum cardinality of a semiglobal cototal dominating set of G and is denoted by*** $\gamma_{sgcot}(G)$***. In this paper we initiate a study of this new parameter*** $\gamma_{sgcot}(G)$***. Some bounds on this parameter are obtained and their exact values for some standard graphs are established.***

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1. Introduction

All graphs considered in this paper are simple, finite, undirected and connected. For all graph theoretic terminology not defined here, the reader is referred to [1]. For a comprehensive introduction to theoretical and applied facts of domination in graphs the reader is directed to the book [2].

A set D of vertices is called a dominating set of G if each vertex not in D is joined to some vertex in D. The domination number γ(G) iis the minimum cardinality of the dominating set of G [2].

A dominating set D is called a global dominating set of G if D is a dominating set of both G and G^c. The global domination number of G, denoted by $\gamma_g(G)$ is the minimum cardinality of the global dominating set of G [6]. A dominating set D of a connected graph G is called a connected dominating set of G if the induced subgraph <D> is connected. The connected domination number of G, denoted by $\gamma_c(G)$ is the minimum cardinality of the connected dominating set of G [7]. A dominating set D of a connected graph is called a independent dominating set of G if the induced subgraph <D> is a null graph [2].

G be a connected graph, then the semi complementary graph of G, denoted by G^{sc}, has the same vertex set as that of G and has edge set {uv / u,v ∈ V(G), uv $\notin E(G)$ and there is w ∈ V(G) such that uw,wv ∈ E(G)} [4]. Let G be a connected graph. Then G is said to be semi complete if any pair of vertices in G have a common neighbor. The necessary and sufficient condition for a connected graph to be semicomplete is that any pair of vertices lie on the same triangle or lie on two triangles having a common vertex [4].

Recently Nicholas and Sheeba Helen introduced the concept global cototal domination number in [5]. A dominating set D of a graph G is a global cototal dominating set if D is both a global dominating set and a cototal dominating set. The global cototal domination number $\gamma_{gcot}(G)$ is the minimum cardinality of a global cototal domination set of G.

A new graph parameter, the semiglobal domination number, for a connected graph G was introduced by Siva Rama Raju and Kumar Addagarla in [9]. A dominating set D ⊆ V(G) is called a semiglobal dominating set(sgd-set) of G, if D is a dominating set for G and G^{sc}. The semiglobal domination number is the minimum cardinality of a semiglobal dominating set of G and is denoted by $\gamma_{sg}(G)$.

The purpose of this paper is to study the semiglobal cototal domination number, for a connected graph G. We call D ⊆ V(G) a semiglobal cototal dominating set, if D is a dominating set for G and G^{sc} and <V-D> has no isolated vertices in G, where G^{sc} is the semicomplementary graph of G. The semiglobal cototal domination number is the minimum cardinality of a semiglobal cototal dominating set of G and is denoted by $\gamma_{sgcot}(G)$.

In this paper some bounds on this parameter are obtained and their exact values for some standard graphs like cycle, path, complete graph, complete bipartite graph, star and crown graph are established.

2. Main Results

Proposition 1.1 For the cycle C_n, n ≥ 6

$$\gamma_{sgcot}(C_n)=\begin{cases}\frac{n}{3}, & n\equiv 0 \pmod 3;\\ \left\lceil\frac{n}{3}\right\rceil, & n\equiv 1 \pmod 3;\\ \left\lceil\frac{n}{3}\right\rceil+1, & n\equiv 2 \pmod 3.\end{cases}$$

Proof: Let $V(C_n) = \{ v_0, v_1, v_2, v_3, \dots, v_{n-1} \}$ and $E(C_n) = \{ v_i v_{i+1} / i = 0, 1, 2, \dots, n-1\}$, subscript modulo n.

If G = C_n, (n ≥ 6) then $C_n^{sc} = \begin{cases} C_{\frac{n}{2}} \cup C_{\frac{n}{2}} & if\ n\ is\ even;\\ C_n & if\ n\ is\ odd.\end{cases}$

Let D be a minimal semi global cototal dominating set of C_n. Let v_i ∈ D. Then $i+3 \leq n$ is the least positive integer such that v_i, v_{i+3} ∈ D. Hence the result follows.

Proposition 1.2 $\gamma_{sgcot}(K_n) = n,\ n \geq 3$.

Proof: All the vertices are isolated in the semicomplementary graph of the complete graph. Therefore, the semiglobal cototal dominating set must contain all the vertices of K_n.

Hence $\gamma_{sgcot}(K_n) = n,\ n \geq 3$.

Proposition 1.3 $\gamma_{sgcot}(K_{m,n}) = 2\ \ m, n \geq 2$.

Proof: Let V_1 and V_2 be the partite sets of $K_{m,n}$ with $|V_1|$=m , $|V_2|$=n. Every vertex in a partite set dominates every other vertex of the other. Then D = { u_i, v_j } is a minimal dominating set for $K_{m,n}$ where $u_i \in V_1$, $v_j \in V_2$, for some i and j. Then the induced subgraph <V-D> has no isolated vertex. The semicomplementary graph of the complete bipartite graph $K_{m,n}$ is a disconnected graph $K_m \cup K_n$, where $\langle V_1 \rangle = K_m$ and $\langle V_2 \rangle = K_n$. Any two vertices in V_1 or that of V_2 are adjacent in $K_{m,n}^{sc}$. Hence D = { u_i, v_j } is a dominating set for the semicomplementary graph of $K_{m,n}$. Thus $\gamma_{sgcot}(K_{m,n}) = 2,\ m, n \geq 2$.

Proposition 1.4 $\gamma_{sgcot}(K_{1,n}) = n+1,\ n \geq 3$.

Proof: Let V(G) = { v, v_1, v_2, v_3, . . . , v_n }, where v is the only vertex of degree n and each v_i is a pendant vertex adjacent to v. The semicomplementary graph of $K_{1,n}$ is a disconnected graph $K_1 \cup K_n$. Hence D = V(G) is the semiglobal cototal dominating set of $K_{1,n}$.

Thus $\gamma_{sgcot}(K_{1,n}) = n+1,\ n \geq 3$.

The crown graph $C_n \odot K_1$ is the graph obtained from cycle C_n by attaching a pendant edge to each vertex of the cycle.

Proposition 1.5 $\gamma_{sgcot}(C_n \odot K_1) = n$, where n is the length of the cycle.

Proof: Let G = $C_n \odot K_1$. V(G) = { v_0, v_1, v_2, . . . , v_{n-1}} $\cup$ { u_0, u_1, u_2, . . . , u_{n-1} }.
E(G) = { $v_i v_{i+1}$ / i = 0, 1, 2, . . . , n-1, subscript modulo n} $\cup$ { $u_i v_i$ / i = 0, 1, 2, . . . , n-1}. Since the minimal semiglobal cototal dominating set D consists of all the n vertices of K_1 in $C_n \odot K_1$, < V-D > = C_n, has no isolated vertex. Thus $\gamma_{sgcot}(C_n \odot K_1) = n$.

Proposition 1.6 For a path P_n on n vertices,

$$\gamma_{sgcot}(P_n) = \begin{cases} \frac{n}{3}+2, & n \equiv 0 \pmod 3; \\ \frac{n+2}{3}, & n \equiv 1 \pmod 3; \\ \left\lceil \frac{n}{3} \right\rceil + 1, & n \equiv 2 \pmod 3). \end{cases}$$

Proof: Let P_n be the path of order n. V(P_n) = { v_0, v_1, v_2, v_3, . . . , v_{n-1}}.
If G = P_n (n $\geq$ 3) then G^{sc} = $P_{\frac{n}{2}} \cup P_{\frac{n}{2}}$ if n is even

= $P_{\frac{n+1}{2}} \cup P_{\frac{n-1}{2}}$ if n is odd

Since D is a semiglobal cototal dominating set in G, $i+3 \leq n$ is the least positive integer such that v_i, $v_{i+3} \in$ D. D must contain v_0 and v_{n-1}, the end vertex of P_n.

If $n \equiv 0 \pmod 3$ then D contains v_{3i} where i = 0, 1, . . . , $\frac{n-3}{3}$ and the vertices v_{n-2} and v_{n-1}. Hence $|D| = \frac{n-3}{3}+1+2 = \frac{n}{3}+2$.

If $n \equiv 1 \pmod 3$ then D contains v_{3i} where i = 0, 1, . . . , $\frac{n-1}{3}$. Thus $|D| = \frac{n-1}{3}+1 = \frac{n+2}{3}$.

If $n \equiv 2 \pmod 3$ then D has v_{3i} where i = 0, 1, . . . , $\frac{n-2}{3}$ and also v_{n-1}.
In this case $|D| = \frac{n-2}{3}+1+1 = \frac{n+1}{3}+1 = \left\lceil \frac{n}{3} \right\rceil + 1$

Hence the result follows.

Theorem1.7 If G is a connected graph and $\gamma_{sgcot}(G) = 2$, then
(i)There is an edge uv in G such that each vertex inV-{u,v} is adjacent to u or v but not both.
(ii) There is a path P_4, each vertex in V-V(P_4) lies on an edge whose end vertices are dominated by end vertices of P_4.

Proof: Suppose that $\gamma_{sgcot}(G) = 2$. Without loss of generality assume that D = { u, v } is a minimal semiglobal cototal dominating set in G.

Case (i) < D > is connected in G.

Clearly uv is an edge in G. If any vertex w in V - { u, v } is adjacent to both u and v, then D is not a dominating set for G^{sc}. Hence (i) holds.

Case (ii) < D > is not connected in G.

No vertex in V-D is adjacent to both u and v. Hence there is a path P_4 from u to v in G, say uv_1v_2v Let $v_3 \in$ V-V(P_4). Since D is a γ_{sgcot} set in G, v_3 is adjacent to u or v in G but not both. For v_3 to be dominated by a vertex in D, v_3, v are to be connected by a path of length two in G, say v_3v_4v. Hence v_3 lies on an edge v_3v_4 and v_3, v_4 are dominated by u and v (end vertices in P_4) respectively. Hence (ii) holds

Note: Every semiglobal cototal dominating set for G is a global cototal dominating set for G, but the converse is not true. That is, $\gamma_{gcot}(G) \leq \gamma_{sgcot}(G)$.

Similarly, every semiglobal cototal dominating set for G is also a semiglobal dominating set for G, but the converse is not true. That is, $\gamma_{sg}(G) \leq \gamma_{sgcot}(G)$.

Theorem1.8 Let E' be the set of all independent edges of K_n. Then

(i) $\gamma_{sgcot}(K_n - E') = \frac{n+1}{2}$ when n is odd.

(ii) $\gamma_{sgcot}(K_n - E') = \frac{n}{2}$ when n is even.

Proof: Let K_n be a complete graph with n vertices. Let E' be the set of all independent edges.

Case(i) n is odd.

Without loss of generality, let E' contain the independent edges from the outer cycle of K_n. Obviously $|E'| = \frac{n-1}{2}$. Then the semicomplementary of this graph contains $|E'|$ number of K_2's and an isolated vertex. Any one end vertex of each K_2 and the isolated vertex form a semiglobal cotottal dominating set

Hence $\gamma_{sgcot}(K_n - E') = |E'| + 1 = \frac{n-1}{2} + 1 = \frac{n+1}{2}$ when n is odd.

Case(ii) n is even.

Here $|E'| = \frac{n}{2}$. Then the semicomplementary of this graph contains a perfect matching, namely $|E'|K_2$. Any one end vertex of each K_2 form a semiglobal cotottal dominating set.

Hence $\gamma_{sgcot}(K_n - E') = |E'| = \frac{n}{2}$ when n is even.

Theorem1.9 If $G = K_n - \{e\}$, then $\gamma_{sgcot}(G) = n$.

Proof: Let $G = K_n - \{e\}$. Then G^{sc} is a disconnected graph containing (n-2) $K_1 \cup K_2$. Then any global dominating set will contain atleast n-1 vertices (n-2 isolated vertices and one end vertex of K_2). If D is a semi global cotottal dominating set, then $|D| \geq$ n-1. If $|D| =$ n-1, then D contradicts the cotottal domination property that <V-D> has no isolated vertex. Hence $|D| >$ n-1. Hence the theorem.

Theorem1.10 For a semicomplete graph G, $\gamma_{sgcot}(G) \geq 3$.

Proof: Suppose the claim does not hold. Let $\gamma_{sgcot}(G) = 2$
Let D = { v_1, v_2 } be a sgcot-d set in G.

Case(i) <D> is connected in G .
Then v_1v_2 is an edge in G. By the nature of semicomplete graph there is a v_3 in G such that <$v_1v_2v_3$> is a triangle in G. This implies D is not a dominating set in G^{sc} which is a contradiction to D is a sgcot d-set in G.

Case(ii) <D> is disconnected in G .Since G is semicomplete there is v_3 in G such that <$v_1v_3v_2$> is a path in G. Then in G^{sc}, v_3 is not dominated by any vertex in D, a contradiction to D is a sgcot d-set in G. Hence in either case, we get a contradiction to D being a sgcot d- set in G. Since a semicomplete graph has at least a triangle $\gamma_{sgcot}(G) \neq 1$. This implies $\gamma_{sgcot}(G) \geq 3$.

Theorem1.11 Let G be a connected graph such that $\delta(G) \geq 2$ and D is an independent sgcot d–set for G. If there exists an independent set D_1 in V-D then D_1 is a sgcot d-set for G.

Proof: Assume that D_1 is independent. Let $v \in D$. This implies that there is v_1 in D_1 such that vv_1 is in G (since $\delta(G) \geq 2$). Since v_1 is in D_1 and D is independent sgcot set in G there is a v_2 in D, v_3 in V such that $v_1 v_2 v_3$ is a path in G. Clearly $v_3 \in D_1$. Since D_1 is independent, $vv_1v_2v_3$ is a path in G. and vv_3 is not an edge in G. For $v \in D$, there exists $v_1 \in D_1$ such that vv_1 is in G and vv_3 is in G^{sc}. Since v is arbitrary, D_1 is sgcot d- set in G.

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