HOPF Bifurcation Analysis of a Food Web Consisting of Two Logistic Prey and a Harvesting Predator with Modified Leslie- Gower scheme

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Abstract: This paper deals with the dynamics of food web consisting of two logistic preys and a variable harvesting predator of Modified Leslie- Gower type. The existence of Hopf bifurcation analysis with varying key parameter is investigated.

Keywords: Harvesting food web model, Hopf bifurcation, numerical simulation.

1. Introduction

The Lotka-Volterra type predator-prey model is the usual model on which multi-species food web and food chain built in which predator takes food as prey species. It is more difficult to study the food web and food chain to be biologically feasibility. In underlying food web the modified Leslie-Gower type predator getting food as Holling’s type second functional response. The predator population is subjected to exploitation under the catch-per-unit-effort hypothesis \( h_1 X_3 \), harvest rate is being used as control effort of dynamics of food web. The Hopf bifurcation refers to the change in the stability as key parameter is varied. In research paper [5], this mathematical model is studied without harvesting in predator.

2. The Mathematical Model

The Mathematical model for the food web consisting two logistic prey and one Modified Leslie- Gower predator with harvesting effort being proportional to predator density is given by the following non-linear system of equations:

\[
\begin{align*}
\frac{dX_1}{dt} &= r_1 X_1 (1 - \frac{X_1}{K}) - \frac{A_1 X_1 X_1}{1 + B_1 X_1 + B_2 X_2}, \\
\frac{dX_2}{dt} &= r_2 X_2 (1 - \frac{X_2}{K}) - \frac{A_2 X_2 X_2}{1 + B_1 X_1 + B_2 X_2}, \\
\frac{dX_3}{dt} &= r_3 X_3^2 (1 - \frac{1}{S_3 + S_1 X_1 + S_2 X_2}) - h_1 X_3
\end{align*}
\]

Where \( X_i \geq 0, i = 1,2 \) represent the population density of two preys and \( X_3 > 0 \) is the population density of the predator. The constants \( K, r_i, A_i, B_i \) and \( S_i \), are model parameters assuming only positive values. The term \( h_1 X_3 \) indicates the harvesting in the predator population. The constant \( h_1 \) is the harvesting effort. The model does not consider any direct competition between the two prey populations, but they are in apparent competition through the shared predation. Indeed, this apparent competition appears, as both prey types are included in predators diet. In the model, the third equation is written according to the Leslie- Gower scheme in which the conventional carrying capacity term is being replaced by the renewable resources for the predator as \( S_1 X_1 + S_2 X_2 \). The additional constant \( S_3 \) normalizes the residual reductions in the predator population in case of severe scarcity of food. Further, the square term signifies the fact that mating frequency is proportional to the number of males as well as that of females.

Rescaling model we get non-dimensionalised form:

\[
\begin{align*}
\frac{dy_1}{dt} &= y_1 (1 - y_1 - \frac{w_1 y_3}{1 + w_2 y_1 + w_3 y_2}) = y_1 f_1(y_1, y_2, y_3) \\
\frac{dy_2}{dt} &= y_2 [(1 - y_2)w_4 - \frac{w_5 y_3}{1 + w_6 y_2 + w_7 y_1}] = y_2 f_2(y_1, y_2, y_3) \\
\frac{dy_3}{dt} &= w_8 y_3 (1 - \frac{w_9}{1 + \alpha_1 w_2 y_1 + \alpha_2 w_3 y_2}) - w_{10} y_3 = y_3 f_3(y_1, y_2, y_3)
\end{align*}
\]

\[
\begin{align*}
w_i > 0, i = 1,2,3,4,5,6,7 ; &\quad y_i \geq 0, i = 1,2,3 ; \alpha_i > 0, i = 1,2.
\end{align*}
\]
\[ t = r \tau, \quad y_1 = X_1 / K, \quad y_3 = X_3 / K, \quad w_1 = A_1 K / r_1, \quad w_2 = B_1 K, \quad w_3 = B_2 K \]
\[ w_4 = r_2 / r_1, \quad w_5 = A_2 K / r_1, \quad w_6 = r_2 K / r_1, \quad w_7 = 1 / S_3, \quad w_8 = S_1 K / S_3, \quad w_9 = S_2 K / S_3, \]
\[ w_{10} = h_1 / K \]
In the above, for simplicity, we have further assumed that \( B_i \) and \( S_i \), \( i = 1, 2 \) are in same proportion i.e. \( w_8 = \alpha_1 w_2, \quad w_9 = \alpha_2 w_2 \).

3. The existence of Positive Equilibrium Point:

The existence of positive equilibrium point is established in the following theorem:

\[ y_2 = \frac{w_5}{w_1 w_4} \left( y_1 - (1 - \frac{w_1 w_4}{w_5}) \right) > 0 \Rightarrow y_1 - (1 - \frac{w_1 w_4}{w_5}) > 0 \Rightarrow w_5 > w_1 w_4 \quad (5) \]
\[ \hat{y}_3 = \frac{w_{10}}{w_6} \left( 1 + \alpha_1 w_2 \hat{y}_1 + \alpha_2 w_3 \hat{y}_2 - w_7 \right) > 0 \text{ when } 1 + \alpha_1 w_2 \hat{y}_1 + \alpha_2 w_3 \hat{y}_2 > w_7 \]

Since \( 0 < \hat{y}_1 < 1, \quad 0 < \hat{y}_3 < 1 \), therefore, the system will have a positive equilibrium point.
Thus the system will have a positive nonzero solution under condition (3) in this case.
This proves the theorem.

\[ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} -\hat{y}_1 + \frac{w_1 w_4 \hat{y}_1}{w_5} & \frac{w_1 w_4 \hat{y}_1}{w_5} & \frac{w_1 w_4 \hat{y}_1}{w_5} \\
 \frac{w_1 w_4 \hat{y}_1}{w_5} & -\hat{y}_2 + \frac{w_2 w_5 \hat{y}_2}{w_3} & \frac{w_2 w_5 \hat{y}_2}{w_3} \\
 \frac{w_1 w_4 \hat{y}_1}{w_5} & \frac{w_2 w_5 \hat{y}_2}{w_3} & -\hat{y}_3 + \frac{w_6 \hat{y}_3}{w_7} \end{bmatrix} \]

The characteristic equation of variational matrix is
\[ \lambda^3 + a_0 \lambda^2 + a_1 \lambda + a_2 = 0 \quad (6) \]
where
\[ a_0 = -(a_{11} + a_{22} + a_{33}; \quad a_1 = (a_{11} a_{33} + a_{22} a_{33} + a_{12} a_{21} - a_{13} a_{23} - a_{13} a_{31}) \]
\[ a_2 = (a_{13} a_{21} + a_{23} a_{31} + a_{11} a_{23} a_{32} - a_{12} a_{21} a_{32} - a_{13} a_{21} a_{32} - a_{13} a_{22} a_{33}) \]

Let \( a_{11} = -m_1; \quad a_{22} = -m_2; \quad a_{33} = -m_3; \quad a_{13} = -m_4; \quad a_{32} = -m_5. \) Since
\[ a_0 = m_1 + m_2 + m_3 > 0; \quad a_1 = (m_1 m_3 + m_2 m_3 + m_2 m_3 - a_{12} a_{21} + m_4 a_{32} + m_4 a_{31}) > 0; \]
\[ a_2 = (m_1 a_{12} m_3 + m_3 a_{12} m_1 - a_2 m_3 a_{12} + m_4 a_{32} a_{31} + m_4 a_{32} a_{32} + m_2 m_3) > 0. \]

Then applying Routh’s criteria \( a_0 > 0 \) provided \( (a_{11} + a_{22} + a_{33}) < 0 \), that is, \( a_1 < 0, \quad a_2 < 0, a_3 < 0 \).
Also \( a_2 > 0, a_3 > 0 \) and \( a_1 a_3 - a_2 > 0 \). Therefore positive nonzero equilibrium point is locally asymptotically stable.

None of the roots of equation (3) is zero as \( a_0 \neq 0 \).

Substituting \( \hat{\lambda} = \pm i \omega \) into (3), the real and imaginary partitions of the results lead to the following conditions: (i) \( \omega = \pm \sqrt{a_1} \); (ii) \( a_0 \omega^2 = a_2 \);
(i) and (ii) and (3) results that a pair of purely imaginary roots \( \pm i \sqrt{a_1} \) and a real root “ \(-a_0\)”.

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Transversality condition: Let the characteristic equation be such that it contains a real root, say \( c_1 \), and a pair of purely imaginary roots \( \lambda_1, \lambda_2 \):

\[
(\lambda - \lambda_1)(\lambda - \lambda_2) = 0.
\]

Or

\[
\lambda^2 - (2\lambda_1 + c_1)\lambda + (\lambda_1^2 + 2\lambda_1 c_1) = 0.
\]

Comparing the coefficients of (3) and (4) gives

\[
a_1(-a_0 - 2\lambda_1) = -a_2 + 2\lambda_1(2\lambda_1 + a_0)^2 \quad (8)
\]

Differentiating (5) with respect to bifurcation parameter \( w_\gamma \), and substituting \( w_\gamma^* \) and \( \lambda_1 \left( w_\gamma^* \right) = 0 \) yields:

\[
\frac{\partial \lambda_1}{\partial w_\gamma} \bigg|_{w_\gamma = w_\gamma^*} = \frac{\partial a_0}{\partial a_1} + a_1 \frac{\partial a_0}{\partial w_\gamma} - \frac{\partial a_2}{\partial w_\gamma} \quad (9)
\]

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\[
(\lambda - \lambda_1)\left( \lambda - \lambda_2 \right) = 0.
\]

Thus the transversality condition is satisfied. So there exists a family of periodic solutions bifurcating from non zero equilibrium in the neighborhood of \( w_\gamma^* \), that is, the Hopf bifurcation will occur when \( w_\gamma \in (w_\gamma^* - \delta, w_\gamma^* + \delta) \).

5. Numerical Simulation for HOPF bifurcation

In this section, the numerical analysis of underlying harvested food web is carried out under the biological feasible conditions. It is observed that numerically the other parameters are fixed at biologically feasible values. Only key parameter \( w_\gamma \) is varied.

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\frac{\partial \lambda_1}{\partial w_\gamma} \bigg|_{w_\gamma = w_\gamma^*} = \frac{a_0}{2(a_0^2 + a_1^2)} \quad (9)
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6. Conclusion

It is concluded that for a set of parameter values with varying key parameter food web harvesting model shows hopf bifurcation. In this paper hopf bifurcation analysis of food web harvesting model is shown analytically as well as numerically.

References
