Fractional Calculus of Saigo Fractional Integral Operator on Univalent and P-Valent Function

Harish Nagar[1], Seema Kabra[2]

Department of Mathematics, Mewar University, Gangrar,Chittorgarh (Raj), India

Abstract: ***In this paper, we apply the fractional calculus technique for the class $UT(n,p\,\beta,\eta,\delta)$ and $Q_m(\alpha,\gamma,\mu)$ which consists of p-valent and univalent functions with Negative coefficients by using the definition of fractional differentiation, and Integration, we establish four distortion theorems. The theorems 1 to 4 are established for the Saigo Fractional Integral Operator $I_{0,z}^{\beta,\eta,\delta}$ applying on the classes of p-valent functions.***

Keywords: Fractional Calculus, Univalent Function, p-valent function, Special Function, Saigo Fractional Integral Operator.

1. Introduction and Definitions

In this paper, we apply the fractional calculus technique for the class $UT(n,p\,\beta,\eta,\delta)$ and $Q_m(\alpha,\gamma,\mu)$ which consists of p-valent and univalent functions with Negative coefficients by using the definition ofSaigo Fractional Integral Operator, we establish four distortion theorems here.

1.1 Definitions:

(i) Saigo's Fractional Integral Operator:

The fractional integral operator $I_{0,x}^{\alpha,\eta,\delta}$ is defined by [2, p. 2810, eq. (3.4)]

$$I_{0,x}^{\alpha,\eta,\delta} f(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\ {}_2F_1\left(\alpha+\eta, -\delta;\ \alpha; 1-txftdt,\right. \quad \ldots(1.1.1)$$

Here$f(x)$is an analytic function and ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function.

(ii) Gauss hypergeometric function:

The Gauss hypergeometric function**d**efined as follows[4, p.2, eq. (1.1.3)]:

$${}_2F_1(a,b;c;z) = {}_2F_1\left[\begin{matrix} a,\ b\ ; \\ c\quad ; \end{matrix}\ z\right] = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k}\,\frac{z^k}{k!},\quad |x|<1 \quad \ldots(1.1.2)$$

(iii) Univalent and p-Valent Functions

Let S denote the class of function [2, p. 2807, eq. (1.1)]

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \ldots(1.1.3)$$

Which are analytic and univalent in $U = \{z: |z| < 1\}$.

Let T denotes the subclass of S consisting of functions of the form

[2, p. 2808, eq. (1.4)]

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k\ ,\quad (a_k \geq 0) \quad \ldots(1.1.4)$$

A function $f$of T is in

$Q_m(\gamma), (0 \leq \gamma < 1, m \in N_0 = NU\{0\}, N = \{1.2, \ldots.\})$ If

[2, p. 2808, eq. (1.5)]

$$Re\,\left(D^m f(z)\right)^1 > \gamma, \qquad z \in U \quad \ldots(1.1.5)$$

Where $D^m f(z)$denotes usual m[th] order derivative introduced by Rusheweyh.

The class $Q_m(\gamma)$was introduced and studied by Uralegaddi and Sarangi [5].

We aim to study the class $Q_m(\alpha,\gamma,\mu)$which consists of functions$f \in T$, and satisfied the conditions

$$\left|\frac{\{(D^m f(z))^1 - 1\}}{\alpha(D^m f(z))^1 + (1-\gamma)}\right| < \mu, \qquad z \in U \quad \ldots(1.1.6)$$

For $0 \leq \gamma < 1,\ 0 \leq \alpha < 1, 0 < \mu < 1$.

Let S(n, p) be the class of functions $f(z)$of the form [1, p. 1090, eq. (1.1)]

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \qquad n,p \in N = \{1,2,3,\ldots\} \quad \ldots(1.1.7)$$

And

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \qquad (a_k \geq 0;\ n,p \in N = \{1,2,3,\ldots\}) \quad \ldots(1.1.8)$$

This is analytic p-valent in the open unit disk $U = \{z \in c: |z| < 1\}$

Let S*WA (n,p,β) denotes the subclass of S (n, p) consisting of p-valent star like functions of order$\beta, 0 \leq \beta \leq p$, if it also satisfies the inequality

[1, p. 1090, eq. (1.2)]

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \quad z \in U \quad \ldots(1.1.9)$$

And also, let CWA (n,p,β) denotes the subclass of S (n, p) consisting of p-valent star like functions of order$\beta, 0 \leq \beta \leq p$, if it also satisfies the inequality (for more details see [1, p. 1090, eq. (1.3)])

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta, \quad z \in U \quad \ldots(1.1.10)$$

Then, we see that $f(z) \in \mathrm{CWA}(n,p,\beta)$ if and only if $zf'(z) \in S^*\mathrm{WA}(n,p,\beta)$.

2. Results Required

The following results are required here

The following Saigo Fractional Integral Operatorformulas are also required here

Let $\beta >, k > \eta - \delta - 1, then$ [2, p. 2811]

$$I_{0,z}^{\beta,\eta,\delta} z^k = \frac{\Gamma(k+1)\Gamma(k-\eta+\delta+1)}{\Gamma(k-\eta+1)\Gamma(k+\beta+\delta+1)} z^{k-\eta} \quad \ldots(2.1)$$

The following two lemmas for the class of functions $Q_m(\alpha,\gamma,\mu)$ and $UT(n,p,\beta,\eta,\delta)$are also required.

Lemma: 1 [2, p. 2808, eq. (2.1)]:

Let the function f be defined by (2.2.2). then$f \varepsilon Q_m(\alpha,\gamma,\mu)$if and only if. $\sum_{k=n+p}^{\infty} k(1+\mu\alpha)\delta(m,k)a_k \le \mu(\alpha+(1-\gamma))$ …(2.2)

Where $0 \le \alpha < 1, 0 \le \gamma < 1, 0 < \mu \le 1$ and $m \in N_{0,}(m=n)$

$$\delta(m,k) = \binom{m+k-1}{m} \quad \ldots(2.3)$$

$$\delta(m,k) = \frac{\Gamma(m+k)}{\Gamma(m+1)\Gamma(k)} \quad \ldots(2.4)$$

The result (2.2.11) is sharp for the function

$$f(z) = z^p - \frac{\mu(\alpha+(1-\gamma))}{k(1+\mu\alpha)\delta(m,k)} z^k, k \ge n+p \quad \ldots(2.5)$$

Lemma 2 [5, p. 72, eq. (3.16), p. 64, eq. (1.2)]:

A function $f(z) \in UT(n,p,\beta,\eta,\delta) for\ \gamma(-1 \le \gamma < 1 and\ k(k\ge 0)$ if and only if $k=n+p\infty\sigma(\gamma,m,k)ak\le k=n+p\infty\sigma(\gamma,m,k)ak\le 1-\gamma$ …(2.6)

The result is sharp for

$$f(z) = z^p - \frac{(1-\gamma)}{\sigma(\gamma,m,k)} z^k, k \ge n+p \quad \ldots(2.7)$$

3. Main Results

The following four distortion theorems for p-valent function concerning to Saigo Fractional Integral Operatorare established as the main results.

Theorem -1: Let the function $f(z)$is a p-valent function and defined in the class $Q_m(\alpha,\gamma,\mu)$, then we have

$$\left|I_{0,z}^{\beta,\eta,\delta} f(z)\right| \ge \frac{\Gamma(p+1)\Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)} |z|^{p-\eta}$$

$$\left[1 - \frac{\Gamma(n+p+1)\Gamma(n+p-\eta+\delta+1\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1))\mu(\alpha+(1-\gamma))\Gamma(m+1)\Gamma(n+p)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)\Gamma(m+n+p)(1+\mu\alpha)(n+p)} |z|^n\right] \ldots(3.1)$$

For $z \epsilon U_0$ where $U_0 = \begin{Bmatrix} U, & \eta \le 1 \\ U-\{0\}, & \eta > 1 \end{Bmatrix}$

the result which sharp for the function $f(z)$ given by the following form

$$f(z) = z^p - \frac{\Gamma(n+p+1)\Gamma(n+p-\eta+\delta+1\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1))\mu(\alpha+(1-\gamma))\Gamma(m+1)\Gamma(n+p)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)\Gamma(m+n+p)(1+\mu\alpha)(n+p)} z^{n+p} \quad \ldots(3.2)$$

Theorem-2:Let the function $f(z)$is a p-valent function and defined in the class $Q_m(\alpha,\gamma,\mu)$, then we have

$$\left|I_{0,z}^{-\beta,\eta,\delta} f(z)\right| \le \frac{\Gamma(p+1)\Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)} |z|^{p-\eta}$$

$$\left[1 + \frac{\Gamma(n+p+1)\Gamma(n+p-\eta+\delta+1\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1))\mu(\alpha+(1-\gamma))\Gamma(m+1)\Gamma(n+p)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)\Gamma(m+n+p)(1+\mu\alpha)(n+p)} |z|^n\right] \ldots(3.3)$$

For $z \epsilon U_0$ where $U_0 = \begin{Bmatrix} U, & \eta \le 1 \\ U-\{0\}, & \eta > 1 \end{Bmatrix}$

the result which sharp for the function $f(z)$ given by the following form

$$f(z) = z^p - \frac{\Gamma(n+p+1)\Gamma(n+p-\eta+\delta+1\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1))\mu(\alpha+(1-\gamma))\Gamma(m+1)\Gamma(n+p)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)\Gamma(m+n+p)(1+\mu\alpha)(n+p)} z^{n+p} \quad \ldots(3.4)$$

Theorem -3: Let the function $f(z)$ is p-valent function and defined in the class $UT(n,p,\beta,\eta,\delta) if\ \{\sigma(\gamma,m,k)\}_{k=n+p}^{\infty}\ \ n,p \in N$ is a non-decreasing sequence, then we have

$$\left|I_{0,z}^{\beta,\eta,\delta} f(z)\right| \ge \frac{\Gamma(p+1)\Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)} |z|^{p-\eta}$$

$$\left[1 - \frac{\Gamma(n+p+1)\Gamma(n+p-\eta+\delta+1\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1))(1-\gamma)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)\sigma(\gamma,m,n+p)} |z|^n\right] \ldots(3.5)$$

For $z \epsilon U_0$ where $U_0 = \begin{Bmatrix} U, & \eta \le 1 \\ U-\{0\}, & \eta > 1 \end{Bmatrix}$

the result which sharp for the function $f(z)$ given by the following form

$$f(z) = z^p - \frac{\Gamma(n+p+1)\Gamma(n+p-\eta+\delta+1\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1))(1-\gamma)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)\sigma(\gamma,m,n+p)} z^{n+p} \quad \ldots(3.6)$$

Theorem -4: Let the function $f(z)$ is p-valent function and defined in the class $UT(n,p,\beta,\eta,\delta) if\ \{\sigma(\gamma,m,k)\}_{k=n+p}^{\infty}\ \ n,p \in N$ is a non-decreasing sequence, then we have

$$\left|I_{0,z}^{\beta,\eta,\delta} f(z)\right| \ge \frac{\Gamma(p+1)\Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)} |z|^{p-\eta}$$

$$\times\left[1 + \frac{\Gamma(n+p+1)\Gamma(n+p-\eta+\delta+1\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1))(1-\gamma)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)\sigma(\gamma,m,n+p)} |z|^n\right]$$

…(3.7)

For $z\epsilon U_0$ where $U_0 = \begin{Bmatrix} U, & \eta \leq 1 \\ U-\{0\}, & \eta > 1 \end{Bmatrix}$

the result which sharp for the function $f(z)$ given by the following form

$$f(z) = z^p - \frac{\Gamma(n+p+1)\Gamma(n+p-\eta+\delta+1\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1))(1-\gamma)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)\sigma(\gamma,m,n+p)} z^{n+p} \quad \ldots(3.8)$$

Proof of Theorem – 1:

To establish the theorem – 1, we first apply the operator $I_{0,z}^{\beta,\eta,\delta}$ on both sides of (1.1.8), we have

$$I_{0,z}^{\beta,\eta,\delta} f(z) = I_{0,z}^{\beta,\eta,\delta} z^p - \sum_{k=n+p}^{\infty} a_k I_{0,z}^{\beta,\eta,\delta} z^k, \qquad a_k \geq 0, n, p \in N$$

On making use of (2.1), we obtain

$$I_{0,z}^{\beta,\eta,\delta} f(z) = \frac{\Gamma(p+1)\Gamma(p-\eta+\delta+1)}{\Gamma(\mathrm{p}-\eta+1)\Gamma(p+\beta+\delta+1)} z^{p-\eta} - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\eta+\delta+1)}{\Gamma(\mathrm{k}-\eta+1)\Gamma(k+\beta+\delta+1)} a_k z^{k-\eta}$$

It can be written in the following form

$$I_{0,z}^{\beta,\eta,\delta} f(z) \frac{\Gamma(\mathrm{p}-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(p+1)\Gamma(p-\eta+\delta+1)} z^{\eta} = z^p - \sum_{k=n+p}^{\infty} a_k z^k \phi(k,\eta,\beta,\delta)$$

Where

$$\phi(k,\eta,\beta,\delta) = \frac{\Gamma(k+1)\Gamma(\mathrm{k}-\eta+\delta+1)\Gamma(\mathrm{p}-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(\mathrm{k}-\eta+1)\Gamma(k+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)} \quad \ldots(3.9)$$

Since for $k \geq n+p, \phi(k,\eta,\beta,\delta)$ is a decreasing function of k.

We have

$$\phi(k,\eta,\beta,\delta) \leq \phi(n+p,\eta,\beta,\delta)$$

i.e.

$$\phi(k,\eta,\beta,\delta) \leq \frac{\Gamma(n+P+1)\Gamma(n+p-\eta+\delta+1)\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)} \quad \ldots(3.10)$$

on setting

$$G(z) = I_{0,z}^{\beta,\eta,\delta} f(z) \frac{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(p+1)\Gamma(p-\eta+\delta+1)} z^{\eta} \quad \ldots(3.11)$$

There for this above equation in view of (3.9), (3.10), reduce in to the following inequality

$$|G(z)| \geq |z|^p - \phi(n+p,\eta,\beta,\delta) \times |z|^{n+p} \sum_{k=n+p}^{\infty} a_k \geq |z|^p - \phi(n+p,\eta,\beta,\delta)|z|^{n+p} a_{n+p} \quad \ldots(3.12)$$

Now from the Lemma – 1 on using (2.2), (2.3), (2.5), we have

$$(n+p)(1+\mu\alpha)\delta(m,n+p) \sum_{k=n+p}^{\infty} a_k \leq \sum_{k=n+p}^{\infty} k\delta(m,k) a_k (1+\mu\alpha) \leq \mu\big(\alpha+(1-\gamma)\big) a_{n+p} \leq \frac{\mu\big(\alpha+(1-\gamma)\big)}{\delta(m,n+P)(1+\mu\alpha)(n+p)}$$

Then on using (2.4), we have

$$a_{n+p} \leq \frac{\mu(\alpha+(1-\gamma))\Gamma(\mathrm{m}+1)\Gamma(\mathrm{n}+\mathrm{p})}{\Gamma(m+n+P)(1+\mu\alpha)(n+p)} \quad \ldots(3.13)$$

The result in (3.12) in view of (3.13) takes the following form

$$|G(z)| \leq |z|^p - \phi(n+p,\eta,\beta,\delta) \frac{\mu\big(\alpha+(1-\gamma)\big)\Gamma(\mathrm{m}+1)\Gamma(\mathrm{n}+\mathrm{p})}{\Gamma(m+n+P)(1+\mu\alpha)(n+p)} |z|^{n+p}$$

Now on using the result in (3.10) and (3.11), we at once arise at the derived result in (3.1) for the function defined in (3.2).

Proof of Theorem – 2:

To establish the theorem – 2, we first apply the operator $I_{0,z}^{\beta,\eta,\delta}$ on both sides of (1.1.7), we have

$$I_{0,z}^{\beta,\eta,\delta} f(z) = I_{0,z}^{\beta,\eta,\delta} z^p + \sum_{k=n+p}^{\infty} a_k I_{0,z}^{\beta,\eta,\delta} z^k, \qquad a_k \geq 0, n, p \in N$$

On making use of (2.1), we obtain

$$I_{0,z}^{\beta,\eta,\delta} f(z) = \frac{\Gamma(p+1)\Gamma(p-\eta+\delta+1)}{\Gamma(\mathrm{p}-\eta+1)\Gamma(p+\beta+\delta+1)} z^{p-\eta} + \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\eta+\delta+1)}{\Gamma(\mathrm{k}-\eta+1)\Gamma(k+\beta+\delta+1)} a_k z^{k-\eta}$$

It can be written in the following form

$$I_{0,z}^{\beta,\eta,\delta} f(z) \frac{\Gamma(\mathrm{p}-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(p+1)\Gamma(p-\eta+\delta+1)} z^{\eta} = z^p + \sum_{k=n+p}^{\infty} a_k z^k \phi(k,\eta,\beta,\delta)$$

Where

$$\phi(k,\eta,\beta,\delta) = \frac{\Gamma(k+1)\Gamma(\mathrm{k}-\eta+\delta+1)\Gamma(\mathrm{p}-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(\mathrm{k}-\eta+1)\Gamma(k+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)} \quad \ldots(3.14)$$

Since for $k \geq n+p, \phi(k,\eta,\beta,\delta)$ is a decreasing function of k.

We have

$$\phi(k,\eta,\beta,\delta) \leq \phi(n+p,\eta,\beta,\delta)$$

i.e.

$$\phi(k,\eta,\beta,\delta) \leq$$

$$\frac{\Gamma(n+P+1)\Gamma(n+p-\eta+\delta+1)\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)} \quad \ldots(3.15)$$

on setting

$$G(z) = I_{0,z}^{\beta,\eta,\delta} f(z)\frac{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(p+1)\Gamma(p-\eta+\delta+1)} z^{\eta} \quad \ldots(3.16)$$

There for this above equation in view of (2.3.42), (2.3.43), reduce in to the following inequality

$$|G(z)| \le |z|^p + \phi(n+p,\eta,\beta,\delta)|z|^{n+p} \times \sum_{k=n+p}^{\infty} a_k \le |z|^p + \phi(n+p,\eta,\beta,\delta)|z|^{n+p} a_{n+p} \quad \ldots(3.17)$$

Now from the Lemma – 1 on using (2.2), (2.3), (2.5), we have

$$(n+p)(1+\mu\alpha)\delta(m,n+p)\sum_{k=n+p}^{\infty} a_k \le \sum_{k=n+p}^{\infty} k\delta(m,k)a_k(1+\mu\alpha) \le \mu\big(\alpha+(1-\gamma)\big)\, a_{n+p} \le \frac{\mu\big(\alpha+(1-\gamma)\big)}{\delta(m,n+P)(1+\mu\alpha)(n+p)}$$

Then on using (2.4), we have

$$a_{n+p} \le \frac{\mu\big(\alpha+(1-\gamma)\big)\Gamma(m+1)\Gamma(n+p)}{\Gamma(m+n+P)(1+\mu\alpha)(n+p)} \quad \ldots(3.18)$$

The result in (3.17) in view of (3.18) takes the following form

$$|G(z)| \le |z|^p + \phi(n+p,\eta,\beta,\delta)\frac{\mu\big(\alpha+(1-\gamma)\big)\Gamma(m+1)\Gamma(n+p)}{\Gamma(m+n+P)(1+\mu\alpha)(n+p)}|z|^{n+p}$$

Now on using the result in (3.15) and (3.16), we at once arise at the derived result in (3.3) for the function defined in (3.4).

Proof of Theorem – 3:

To establish the theorem – 3, we first apply the operator $I_{0,z}^{\beta,\eta,\delta}$ on both sides of (1.1.8), we have

$$I_{0,z}^{\beta,\eta,\delta} f(z) = I_{0,z}^{\beta,\eta,\delta} z^p - \sum_{k=n+p}^{\infty} a_k I_{0,z}^{\beta,\eta,\delta} z^k, \qquad a_k \ge 0, n, p \in N$$

On making use of (2.1), we obtain

$$I_{0,z}^{\beta,\eta,\delta} f(z) = \frac{\Gamma(p+1)\Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)} z^{p-\eta} - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\eta+\delta+1)}{\Gamma(k-\eta+1)\Gamma(k+\beta+\delta+1)} a_k z^{k-\eta}$$

It can be written in the following form

$$I_{0,z}^{\beta,\eta,\delta} f(z)\frac{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(p+1)\Gamma(p-\eta+\delta+1)} z^{\eta} = z^p - \sum_{k=n+p}^{\infty} a_k z^k \phi(k,\eta,\beta,\delta)$$

Where

$$\phi(k,\eta,\beta,\delta) = \frac{\Gamma(k+1)\Gamma(k-\eta+\delta+1)\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(k-\eta+1)\Gamma(k+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)} \quad \ldots(3.19)$$

Since for $k \ge n+p, \phi(k,\eta,\beta,\delta)$ is a decreasing function of k.

We have

$$\phi(k,\eta,\beta,\delta) \le \phi(n+p,\eta,\beta,\delta)$$

i.e.

$$\phi(k,\eta,\beta,\delta) \le \frac{\Gamma(n+P+1)\Gamma(n+p-\eta+\delta+1)\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)} \quad \ldots(3.20)$$

on setting

$$G(z) = I_{0,z}^{\beta,\eta,\delta} f(z)\frac{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(p+1)\Gamma(p-\eta+\delta+1)} z^{\eta} \quad \ldots(3.21)$$

There for this above equation in view of (3.19), (3.20), reduce in to the following inequality

$$|G(z)| \ge |z|^p - \phi(n+p,\eta,\beta,\delta)|z|^{n+p} \sum_{k=n+p}^{\infty} a_k \ge |z|^p - \phi(n+p,\eta,\beta,\delta)|z|^{n+p} a_{n+p} \quad \ldots(3.22)$$

Now from the Lemma – 2 on using (2.6), (2.7), we have

$$\sigma(\gamma,m,n+p)\sum_{k=n+p}^{\infty} a_k \le \sum_{k=n+p}^{\infty} \sigma(\gamma,m,k)a_k \le (1-\gamma)\, a_{n+p} \le \frac{(1-\gamma)}{\sigma(\gamma,m,n+P)} \quad \ldots(3.23)$$

The result in (3.22) in view of (3.23) takes the following form

$$|G(z)| \ge |z|^p - \phi(n+p,\eta,\beta,\delta)\frac{(1-\gamma)}{\sigma(\gamma,m,n+P)}|z|^{n+p}$$
$$m \ge 0, -1 \le \gamma < 1$$

Now on using the result in (3.20) and (3.21), we at once arise at the derived result in (3.5) for the function defined in (3.6).

Proof of Theorem – 4:

To establish the theorem – 4, we first apply the operator $I_{0,z}^{\beta,\eta,\delta}$ on both sides of (1.1.7), we have

$$I_{0,z}^{\beta,\eta,\delta} f(z) = I_{0,z}^{\beta,\eta,\delta} z^p + \sum_{k=n+p}^{\infty} a_k I_{0,z}^{\beta,\eta,\delta} z^k, \qquad a_k \ge 0, n, p \in N$$

On making use of (2.1), we obtain

$$I_{0,z}^{\beta,\eta,\delta} f(z) = \frac{\Gamma(p+1)\Gamma(p-\eta+\delta+1)}{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)} z^{p-\eta} + \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\eta+\delta+1)}{\Gamma(k-\eta+1)\Gamma(k+\beta+\delta+1)} a_k z^{k-\eta}$$

It can be written in the following form

$$I_{0,z}^{\beta,\eta,\delta} f(z) \frac{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(p+1)\Gamma(p-\eta+\delta+1)} z^{\eta} = z^p + \sum_{k=n+p}^{\infty} a_k z^k \phi(k,\eta,\beta,\delta)$$

Where

$$\phi(k,\eta,\beta,\delta) = \frac{\Gamma(k+1)\Gamma(k-\eta+\delta+1)\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(k-\eta+1)\Gamma(k+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)} \quad \ldots(3.24)$$

Since for $k \geq n+p, \phi(k,\eta,\beta,\delta)$ is a decreasing function of k.

We have

$$\phi(k,\eta,\beta,\delta) \leq \phi(n+p,\eta,\beta,\delta)$$

i.e.

$$\phi(k,\eta,\beta,\delta) \leq \frac{\Gamma(n+P+1)\Gamma(n+p-\eta+\delta+1)\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(n+p-\eta+1)\Gamma(n+p+\beta+\delta+1)\Gamma(p+1)\Gamma(p-\eta+\delta+1)} \quad \ldots(3.25)$$

on setting

$$G(z) = I_{0,z}^{\beta,\eta,\delta} f(z) \frac{\Gamma(p-\eta+1)\Gamma(p+\beta+\delta+1)}{\Gamma(p+1)\Gamma(p-\eta+\delta+1)} z^{\eta} \quad \ldots(3.26)$$

There for this above equation in view of (3.24), (3.25), reduce in to the following inequality

$$|G(z)| \leq |z|^p + \phi(n+p,\eta,\beta,\delta)|z|^{n+p}$$

$$\sum_{k=n+p}^{\infty} a_k \leq |z|^p + \phi(n+p,\eta,\beta,\delta)|z|^{n+p} a_{n+p} \quad \ldots(3.27)$$

Now from the Lemma – 2 on using (2.6), (2.7), we have

$$\sigma(\gamma,m,n+p) \sum_{k=n+p}^{\infty} a_k \leq \sum_{k=n+p}^{\infty} \sigma(\gamma,m,k) a_k \leq (1-\gamma)\, a_{n+p} \leq \frac{(1-\gamma)}{\sigma(\gamma,m,n+P)} \quad \ldots(3.28)$$

The result in (3.27) in view of (3.28) takes the following form

$$|G(z)| \geq |z|^p + \phi(n+p,\eta,\beta,\delta) \frac{(1-\gamma)}{\sigma(\gamma,m,n+P)} |z|^{n+p}$$

$$m \geq 0, -1 \leq \gamma < 1$$

Now on using the result in (3.25) and (3.26), we at once arise at the derived result in (3.7) for the function defined in (3.8).

4. Special Cases

(1) If in (3.1) and (3.3), we take n = p = 1, then these results reduce to the following known results [2, p. 2811, eqns. (3.7), (3.8)] i.e.

$$\left|I_{0,z}^{\beta,\eta,\delta} f(z)\right| \geq \frac{\Gamma(2-\eta+\delta)}{\Gamma(2-\eta)\Gamma(2+\beta+\delta)} |z|^{1-\eta} \left[1 - \right.$$

$$2-\eta+\delta\mu\alpha+1-\gamma 2+\beta+\delta(2-\eta)1+\mu\alpha m+1z \ldots(4.1)$$

$$m = n, m, n \in N_0$$

$$\left|I_{0,z}^{\beta,\eta,\delta} f(z)\right| \leq \frac{\Gamma(2-\eta+\delta)}{\Gamma(2-\eta)\Gamma(2+\beta+\delta)} |z|^{1-\eta} \left[1 + \frac{(2-\eta+\delta)\mu(\alpha+(1-\gamma))}{(2+\beta+\delta)(2-\eta)(1+\mu\alpha)(m+1)} |z|\right] \ldots(4.2)$$

$$m = n, m, n \in N_0$$

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