Ultra Semi $g\alpha$-Closed Graphs

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Abstract: In this paper, we introduce the notion of ultra semi $g\alpha$-closed graphs and strongly semi $g\alpha$-closed graphs in topological spaces and investigate some of their properties via semi $g\alpha$-open sets and semi $g\alpha$-closure operator. We also introduce the notion of semi $g\alpha$-Urysohn space and examine its properties.

Keywords: semi $g\alpha$-closed graphs, ultra semi $g\alpha$-closed graphs, strongly semi $g\alpha$-closed graphs, semi $g\alpha$-Urysohn space, semi $g\alpha$-$T_1$ space.

1. Introduction

In 2009, M. Caldas et al. [1] introduced and studied the concept of functions with strongly $\lambda$-closed graphs. V. Kokilavani and M. Vivek Prabu [4], introduced the notion of semi $g\alpha$-closed sets in topological spaces and examined their relationship with the other existing sets. In this paper, we introduce the notion of ultra semi $g\alpha$-closed graphs and strongly semi $g\alpha$-closed graphs in topological spaces and investigate some of their properties via semi $g\alpha$-open sets and semi $g\alpha$-closure operator. We also introduce the notion of semi $g\alpha$-Urysohn space and examine its properties.

2. Preliminaries

Definition 2.1 A subset $A$ of $X$ is called

1) $g$-closed [6] if $cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $X$. The complement of $g$-closed set is called $g$-open.

2) $g\alpha$-closed [7] if $acl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $X$. The complement of $g\alpha$-closed set is called $g\alpha$-open.

3) $g\alpha$-closed [2] if $acl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $X$. The complement of $g\alpha$-closed set is called $g\alpha$-open.

4) semi $g\alpha$-closed [4] if $scl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $g\alpha$-open in $X$. The complement of semi $g\alpha$-closed set is called semi $g\alpha$-open.

Definition 2.2 A function $f : X \rightarrow Y$ is said to be

1) semi $g\alpha$-continuous [4] if for every closed set in $Y$, its inverse image is semi $g\alpha$-closed in $X$.

2) semi $g\alpha$-irresolute [4] if for every semi $g\alpha$-closed set in $Y$, its inverse image is semi $g\alpha$-closed in $X$.

Definition 2.3

1) A space $X$ is said to be semi $g\alpha$-$T_0$ [5] if for each pair of distinct points $x$ and $y$ in $X$, there exists semi $g\alpha$-open sets $U$ and $V$ containing $x$ and $y$ respectively, such that $x \in U$ and $y \notin U$ or $y \in V$ and $x \notin V$.

2) A space $X$ is said to be semi $g\alpha$-$T_1$ [5] if for each pair of distinct points $x$ and $y$ in $X$, there exists semi $g\alpha$-open sets $U$ and $V$ containing $x$ and $y$ respectively, such that $y \notin U$ and $x \notin V$.

3) A space $X$ is said to be semi $g\alpha$-$T_2$ [5] if for each pair of distinct points $x$ and $y$ in $X$, there exists semi $g\alpha$-open sets $U$ and $V$ containing $x$ and $y$ respectively, such that $U \cap V = \emptyset$.

Definition 2.4 If $f : (X,\tau) \rightarrow (Y,\sigma)$ is any function, then the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called graph of $f$ [3].

3. Ultra Semi $g\alpha$-Closed Graphs

Definition 3.1 A function $f : (X,\tau) \rightarrow (Y,\sigma)$ is said to have a ultra semi $g\alpha$-closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \subseteq \text{Semi}^\# GaO(X, x)$ and $V \subseteq \text{Semi}^\# GaO(Y, y)$ such that $(f(U) \cap \text{semi}^\# Ga-cl(V)) = \emptyset$.

Theorem 3.2 If $f : (X,\tau) \rightarrow (Y,\sigma)$ is a function with a ultra semi $g\alpha$-closed graph, then for each $x \in X$, $f(x) = \cap\{\text{semi}^\# Ga-cl(f(U)) : U \subseteq \text{Semi}^\# GaO(X, x)\}$.

Proof. Suppose the theorem is false. Then there exists an $y \neq f(x)$ such that $y \in \cap\{\text{semi}^\# Ga-cl(f(U)) : U \subseteq \text{Semi}^\# GaO(X, x)\}$. Hence for every $U \subseteq \text{Semi}^\# GaO(X, x)$, $y \in \text{semi}^\# Ga-cl(f(U))$. So $V \cap f(U) \neq \emptyset$ for every $V \subseteq Y$. 

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Semi\#GAO(Y,\gamma). This implies that semi \#ga-cl(V) \cap f(U) \nRightarrow V \cap f(U) \neq \emptyset which contradicts the hypothesis that f is a function with a ultra semi \#ga-closed graph. Hence the theorem holds.

Theorem 3.3 If f:(X,\tau)\rightarrow(Y,\sigma) is semi \#ga-irresolute and Y is semi \#ga-T_2, then G(f) is ultra semi \#ga-closed.

Proof. Let (x,y) \in (X \times Y) \setminus G(f) and V \subseteq \text{Semi}\#GAO(Y,\gamma) such that f(x) \notin \text{semi} \#ga-cl(V). It follows that there is U \subseteq \text{Semi}\#GAO(X,\tau) such that f(U) \subseteq Y \setminus \text{semi} \#ga-cl(V). Hence, f(U) \cap \text{semi} \#ga-cl(V) = \emptyset. The converse of the above theorem need not be true which can be seen from the following example.

Example 3.4 Let X = \{a,b,c\}, \tau = \{\emptyset,X,\{a\}\} and f:(X,\tau)\rightarrow(X,\tau) be an identity map. Then clearly f is semi \#ga-irresolute but X is not a semi \#ga-T_2 space. Therefore G(f) is not ultra semi \#ga-closed.

Theorem 3.5 If f:(X,\tau)\rightarrow(Y,\sigma) is surjective and has a ultra semi \#ga-closed graph G(f), then Y is both semi \#ga-T_1 and semi \#ga-T_2.

Proof. Let y_1 \neq y_2 \in Y. Since f is surjective, there exists any x \in X such that f(x_1) = y_1. Now (x_1,y_2) \in (X \times Y) \setminus G(f). The ultra semi \#ga-closed graph G(f) of f implies U \subseteq \text{Semi}\#GAO(X,x_1) and V \subseteq \text{Semi}\#GAO(Y,y_2) such that f(U) \cap \text{semi} \#ga-cl(V) = \emptyset. Therefore there exists any W \subseteq \text{Semi}\#GAO(Y,y_1) such that W \cap V = \emptyset. Thus Y is semi \#ga-T_2 and hence it is a semi \#ga-T_1 space.

Theorem 3.6 If f: (X,\tau)\rightarrow(Y,\sigma) is injective and has a ultra semi \#ga-closed graph G(f), then X is a semi \#ga-T_1 space.

Proof. Since f is injective, for any pair of distinct points x_1,x_2 \in X, f(x_1) \neq f(x_2). Here (x_1,f(x_1)) \in (X \times Y) \setminus G(f). Since G(f) is ultra semi \#ga-closed graph, there exist U \subseteq \text{Semi}\#GAO(X,x_1) and V \subseteq \text{Semi}\#GAO(Y,f(x_1)) such that f(U) \cap \text{semi} \#ga-cl(V) = \emptyset. Therefore there exist any W \subseteq \text{Semi}\#GAO(x_1,x_2) such that f(U) \cap \text{semi} \#ga-cl(V) = \emptyset. Hence, W \subseteq \text{semi}\#ga-T_1 space.

Remark 3.7 If f:(X,\tau)\rightarrow(Y,\sigma) is bijective and has a ultra semi \#ga-closed graph G(f), then X and Y are semi \#ga-T_1 spaces.

Theorem 3.8 A space X is semi \#ga-T_2 if and only if the identity function f: (X,\tau)\rightarrow(X,\tau) has a ultra semi \#ga-closed graph G(f).

Proof. Let X be a semi \#ga-T_2 space. Since the identity function f: (X,\tau)\rightarrow(X,\tau) is semi \#ga-irresolute, from Theorem 3.3 we conclude that it has a ultra semi \#ga-closed graph G(f).

Conversely suppose that f has a ultra semi \#ga-closed graph G(f). Here clearly f is surjective and hence by Theorem 3.5, X is a semi \#ga-T_2 space.

Definition 3.9 A function f: (X,\tau)\rightarrow(Y,\sigma) is called quasi semi \#ga-irresolute, if for each x \in X and each V \subseteq \text{Semi}\#GAO(Y,f(x)), there exist U \subseteq \text{Semi}\#GAO(X,x) such that f(U) \subseteq \text{semi}\#ga-cl(V).

Theorem 3.10 If a function f:(X,\tau)\rightarrow(Y,\sigma) is quasi semi \#ga-irresolute, injective and has a ultra semi \#ga-closed graph G(f), then X is semi \#ga-T_2.

Proof. Since f is injective, for any pair of distinct points x_1,x_2 \in X, f(x_1) \neq f(x_2). Here (x_1,f(x_1)) \notin (X \times Y) \setminus G(f). Since G(f) is ultra semi \#ga-closed graph, there exist U \subseteq \text{Semi}\#GAO(X,x_1) and V \subseteq \text{Semi}\#GAO(Y,f(x_1)) such that f(U) \cap \text{semi} \#ga-cl(V) = \emptyset. Consequently f^{-1}(\text{semi} \#ga-cl(V)) \subseteq X \setminus U. Moreover since f is quasi semi \#ga-irresolute, there exists any W \subseteq \text{Semi}\#GAO(X,x_2) such that f(W) \subseteq \text{semi} \#ga-cl(V). i.e., W \subseteq f^{-1}(\text{semi} \#ga-cl(V)) \subseteq X \setminus U. Thus W \cap U = \emptyset. Hence X is semi \#ga-T_2.

Theorem 3.11 If a function f:(X,\tau)\rightarrow(Y,\sigma) is semi \#ga-irresolute, injective and has a ultra semi \#ga-closed graph G(f), then X is semi \#ga-T_2.

Proof. Since every semi \#ga-irresolute function is quasi semi \#ga-irresolute, the proof follows from Theorem 3.10.

Theorem 3.12 If a function f:(X,\tau)\rightarrow(Y,\sigma) is quasi semi \#ga-irresolute, bijective and has a ultra semi \#ga-closed graph G(f), then X and Y are semi \#ga-T_2.

Proof. It is obvious from Theorem 3.10 and Theorem 3.5.

4. Strongly Semi \#ga-Closed Graphs

Definition 4.1 A function f: (X,\tau)\rightarrow(Y,\sigma) is said to have a strongly semi \#ga-closed graph if for each (x,y) \in (X \times Y) \setminus G(f), there exist U \subseteq \text{Semi}\#GAO(x,x) and an open set V of Y containing y such that f(U) \cap V = \emptyset.

Theorem 4.2 Every ultra semi \#ga-closed graph is...
strongly semi $\# ga$-closed graph.

**Proof.** It follows from the definitions 3.1 and 4.1.

**Theorem 4.3** If $f: (X,\tau)\rightarrow(Y,\sigma)$ is semi $\# ga$-continuous and $Y$ is Hausdorff, then $G(f)$ is strongly semi $\# ga$-closed in $X \times Y$.

**Proof.** Let $(x,y) \in (X \times Y) \setminus G(f)$. Then $f(x) \neq y$. Since $Y$ is Hausdorff, there exist open sets $V$ and $W$ in containing $f(x)$ and $y$ respectively such that $V \cap W = \emptyset$. Also since $f$ is semi $\# ga$-continuous, there exists $U \subseteq \text{Semi}^\# GaO(X,x)$ such that $f(U) \subseteq V$. Hence $f(U) \cap W = \emptyset$, $G(f)$ is strongly semi $\# ga$-closed.

**Theorem 4.4** If $f: (X,\tau)\rightarrow(Y,\sigma)$ is surjective and has a strongly semi $\# ga$-closed graph $G(f)$, then $Y$ is $T_1$.

**Proof.** Let $y_1 \neq y_2 \in Y$. Since $f$ is surjective, there exists a $x \in X$ such that $f(x) = y_2$. Hence $(x,y_1) \notin G(f)$. Then by the definition 4.1, there exist semi $\# ga$-open set $U$ and an open set $V$ containing $x$ and $y_1$ respectively, such that $f(U) \cap V = \emptyset$. Hence $y_2 \notin V$. Thus $Y$ is $T_1$.

**Theorem 4.5** If $f: (X,\tau)\rightarrow(Y,\sigma)$ is a function with a strongly semi $\# ga$-closed graph, then for each $x \in X$, $f(x) = \cap \{\text{Semi}^\# \text{gocl}(f(U)) : U \subseteq \text{Semi}^\# GaO(X,x)\}$.

**Proof.** It follows from the Theorem 3.2 and Theorem 4.2.

**Theorem 4.6** If $f: (X,\tau)\rightarrow(Y,\sigma)$ is surjective and has a strongly semi $\# ga$-closed graph $G(f)$, then $Y$ is both semi $\# ga$-$T_2$ and semi $\# ga$-$T_1$.

**Proof.** It follows from Theorem 3.5 and Theorem 4.2.

**Theorem 4.7** If $f: (X,\tau)\rightarrow(Y,\sigma)$ is an injection and $G(f)$ is strongly semi $\# ga$-closed, then $X$ is semi $\# ga$-$T_1$.

**Proof.** It follows from the Theorem 3.6 and Theorem 4.2.

**Theorem 4.8** If $f: (X,\tau)\rightarrow(Y,\sigma)$ is a bijective function with strongly semi $\# ga$-closed graph $G(f)$, then $(X,\tau)$ and $(Y,\sigma)$ are semi $\# ga$-$T_2$ space.

**Proof.** It follows from Theorem 3.7 and Theorem 4.2.

**Theorem 4.9** If $f: (X,\tau)\rightarrow(Y,\sigma)$ is semi $\# ga$-irresolute and $Y$ is semi $\# ga$-$T_2$, then $G(f)$ is strongly semi $\# ga$-closed.

**Proof.** It follows from the Theorem 3.3 and Theorem 4.2.

**Example 4.10** Let $X = \{a,b,c\}$, $\tau = \{\emptyset,X,\{a\}\}$ and $f: (X,\tau)\rightarrow(X,\tau)$ be an identity map. Then clearly $f$ is semi $\# ga$-irresolute but $X$ is not a semi $\# ga$-$T_2$ space. Therefore $G(f)$ is not strongly semi $\# ga$-closed.

**Theorem 4.11** A space $X$ is semi $\# ga$-$T_2$ if and only if the identity function $f: (X,\tau)\rightarrow(X,\tau)$ has a strongly semi $\# ga$-closed graph $G(f)$.

**Proof.** It follows from the Theorem 3.8 and Theorem 4.2.

**Theorem 4.12** If a function $f: (X,\tau)\rightarrow(Y,\sigma)$ is a quasi semi $\# ga$-irresolute injection with a strongly semi $\# ga$-closed graph $G(f)$, then $X$ is semi $\# ga$-$T_2$.

**Proof.** Since every semi $\# ga$-irresolute function is quasi semi $\# ga$-irresolute, the proof follows from Theorem 3.11.

**Theorem 4.13** If a function $f: (X,\tau)\rightarrow(Y,\sigma)$ is semi $\# ga$-irresolute, injective and has a strongly semi $\# ga$-closed graph $G(f)$, then $X$ and $Y$ are semi $\# ga$-$T_2$.

**Proof.** It is obvious from Theorem 3.12 and Theorem 4.2.

5. **Semi $\# ga$-Urysohn Space**

**Definition 5.1** A topological space $X$ is called semi $\# ga$-Urysohn if every pair of distinct points $x, y \in X$, there exist $U \subseteq \text{Semi}^\# GaO(X,x)$ and $V \subseteq \text{Semi}^\# GaO(X,y)$ such that $\text{semi}^\# ga-cl(U) \cap \text{semi}^\# ga-cl(V) = \emptyset$.

**Theorem 5.2** Every semi $\# ga$-Urysohn space is a semi $\# ga$-$T_2$ space.

**Proof.** Let $x$ and $y$ be two distinct points of $X$. Since $X$ is semi $\# ga$-Urysohn, there exist $U \subseteq \text{Semi}^\# GaO(X,x)$ and $V \subseteq \text{Semi}^\# GaO(X,y)$ such that $\text{semi}^\# ga-cl(U) \cap \text{semi}^\# ga-cl(V) = \emptyset$. Hence $U \cap V = \emptyset$. Thus $X$ is semi $\# ga$-$T_2$.

**Theorem 5.3** If $Y$ is semi $\# ga$-Urysohn and $f: (X,\tau)\rightarrow(Y,\sigma)$ is quasi semi $\# ga$-irresolute injection, then $X$ is semi $\# ga$-$T_2$.
Proof. Since $f$ is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. Also since $Y$ is semi # ga-Urysohn, there exist $V_i \subseteq \text{Semi}#GaO(Y, f(x_i)), i = 1, 2$ such that semi # $\text{ga-cl}(V_i) \cap \text{semi} # \text{ga-cl}(V_2) = \emptyset$. Hence $f^{-1}(\text{semi} # \text{ga-cl}(V_i)) \cap f^{-1}(\text{semi} # \text{ga-cl}(V_2)) = \emptyset$. Since $f$ is quasi semi # $\text{ga}$-irresolute, there exist $U_i \subseteq \text{Semi}#GaO(X, x_i)$, such that $f(U_i) \subseteq \text{semi} # \text{ga-cl}(V_i), i = 1, 2$. Hence $U_i \subseteq f^{-1}(\text{semi} # \text{ga-cl}(V_i)), i = 1, 2$. Therefore $U_1 \cap U_2 \subseteq f^{-1}(\text{semi} # \text{ga-cl}(V_i)) \cap f^{-1}(\text{semi} # \text{ga-cl}(V_2)) = \emptyset$. Thus $X$ is semi # $g_a$-$T_2$.

Definition 5.4 A function $f: (X,\tau) \rightarrow (Y,\sigma)$ is pre semi # $g_a$-open if $f(A) \subseteq \text{Semi}#GaO(Y)$ for all $A \subseteq \text{Semi}#GaO(X)$.

Lemma 5.5 Let $f: (X,\tau) \rightarrow (Y,\sigma)$ be pre semi # $g_a$-open, bijective. Then for any $B \subseteq \text{Semi}#GaC(X)$, $f(B) \subseteq \text{Semi}#GaC(Y)$.

Theorem 5.6 If $f: (X,\tau) \rightarrow (Y,\sigma)$ is pre semi # $g_a$-open, bijective and $X$ is semi # $g_a$-Urysohn, then $Y$ is semi # $g_a$-Urysohn.

Proof. Let $y_1 \neq y_2 \in Y$. Since $f$ is bijective, $f^{-1}(y_1) \neq f^{-1}(y_2) \in X$. Also since $X$ is semi # $g_a$-Urysohn, there exist $U \subseteq \text{Semi}#GaO(X, f^{-1}(y_1))$ and $V \subseteq \text{Semi}#GaO(X, f^{-1}(y_2))$ such that semi # $\text{ga-cl}(U) \cap \text{semi} # \text{ga-cl}(V) = \emptyset$. Since semi # $\text{ga-cl}(U)$ is a semi # $\text{ga-cl}$-closed set in $X$, by Lemma 5.5 we have $f(\text{semi} # \text{ga-cl}(U)) \subseteq \text{Semi}#GaC(Y)$. Also $U \subseteq \text{semi} # \text{ga-cl}(U)$ implies $f(U) \subseteq f(\text{semi} # \text{ga-cl}(U))$ and hence semi # $\text{ga-cl}(f(U)) \subseteq \text{semi} # \text{ga-cl}(f(\text{semi} # \text{ga-cl}(U))) = f(\text{semi} # \text{ga-cl}(U))$. Similarly we have semi # $\text{ga-cl}(f(V)) \subseteq f(\text{semi} # \text{ga-cl}(V))$. Since $f$ is injective, semi # $\text{ga-cl}(f(U)) \cap \text{semi} # \text{ga-cl}(f(V)) \subseteq f(\text{semi} # \text{ga-cl}(U)) \cap f(\text{semi} # \text{ga-cl}(V)) = f(\text{semi} # \text{ga-cl}(U) \cap \text{semi} # \text{ga-cl}(V)) = \emptyset$. Also since $f$ is pre semi # $g_a$-open, there exist $f(U) \subseteq \text{Semi}#GaO(Y, y_1)$ and $f(V) \subseteq \text{Semi} # \text{gaO}(Y, y_2)$ such that semi # $\text{ga-cl}(f(U)) \cap \text{semi} # \text{ga-cl}(f(V)) = \emptyset$. Thus $Y$ is semi # $g_a$-Urysohn.

Theorem 5.7 If $f: (X,\tau) \rightarrow (Y,\sigma)$ is pre semi # $g_a$-open, bijective and $X$ is semi # $g_a$-$T_2$, then $G(f)$ is ultra semi # $g_a$-closed.

Proof. Let $(x,y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since $f$ is bijective, $x \neq f^{-1}(y)$. Also since $X$ is semi # $g_a$-$T_2$, there exist $U_x, U_y \subseteq \text{Semi}#GaO(X)$ such that $x \in U_x$, $f^{-1}(y) \in U_y$ and $U_x \cap U_y = \emptyset$. Moreover as $f$ is pre semi # $g_a$-open and bijective, we have $f(x) \in f(U_x) \subseteq \text{Semi}#GaO(Y)$. $y \in f(U_y) \subseteq \text{Semi}#GaO(Y)$ and $f(U_x) \cap f(U_y) = \emptyset$. Hence $f(U_x) \cap \text{semi} # \text{ga-cl}(f(U_y)) = \emptyset$. Therefore $G(f)$ is ultra semi # $g_a$-closed.

Theorem 5.8 If $f: (X,\tau) \rightarrow (Y,\sigma)$ is quasi semi # $g_a$-irresolute and $Y$ is semi # $g_a$-Urysohn, then $G(f)$ is ultra semi # $g_a$-closed.

Proof. Let $(x,y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$. Since $Y$ is semi # $g_a$-Urysohn, there exist $V \subseteq \text{Semi}#GaO(Y, y)$ and $W \subseteq \text{Semi}#GaO(Y, f(x))$ such that semi # $\text{ga-cl}(V) \cap \text{semi} # \text{ga-cl}(W) = \emptyset$. Since $f$ is quasi semi # $g_a$-irresolute, there exists $U \subseteq \text{Semi}#GaO(X, x)$ such that $f(U) \subseteq \text{semi} # \text{ga-cl}(W)$. Hence we have $f(U) \cap \text{semi} # \text{ga-cl}(V) = \emptyset$. Thus $G(f)$ is ultra semi # $g_a$-closed.

References


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