Scheduling Model for Special Class of Flow Shop Problem

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Abstract: ***The present paper is explored to deal with multi-criteria in scheduling for a special class of job shop problem. We have considered a (n, k) job shop problem in which we have m-parallel service platforms to serve same type of job with different potential and rest others are available to provide particular job services. Our objective is to find out the optimum processing cost, completion time & schedule.***

Keywords: α -cut, Fuzzy Ranking, LPP, LR-type fuzzy number, CDS Algorithm.

1. Introduction

In the present paper deals with a very-special for the flow shop scheduling problem in which there are m-parallel service stations available in place of first regular machine task, it can be explained with the help of example. Suppose we have to complete a job work viz. publication of different type of books with predefined no. of copies. For publication of i th type of book we have to publish a_i number of copies and we have m vendors (publication houses) in the market with different publication costs and different processing time for each i^{th} type of book. Now we have to allot the books publishing work to the vendors such that the work completes in minimum time with minimum processing cost.

Similar type of work has already been reported to some extent by Maggu & Das [17]. They have assumed equi-potential parallel machine with different processing per unit cost under deterministic environment.

In the present work we have considered the case in which all the service stations are of different potentials with different processing cost, different processing time and unequal capacity to do the job and our objective is to find out minimum processing cost, minimum completion time and optimal schedule under certain constraints, taking into consideration of uncertainty to some extent. The problem can be formulated as a linear programming problem and the solution of these type of problem can be determine with the help of fuzzy integer generalized transportation problem given by Omar M. Saad [20], A.K. Bit [4] & Stefan Chanas et all [7]. First obtaining optimal processing cost for the n x m non-equi-potential service stations and n x k job shop problem, then we schedule the n x k job shop problem and determine minimum completion time with the help of methods explained in Kumar P. [13]. In the present work we have introduced a term sub job which can be explained as, suppose a company manufacture a particular model of T.V. and the model represents job and if we manufacture 100 numbers of T.V. of the same model then each T.V. Manufacturing process can be considered as a sub-job for the job, to manufacture the particular model.

2. Preliminaries

Def. 2.1: The α -level set:

The set of elements that belong to the fuzzy set Ã at least to the degree α is called the α *-level set* for LR-type fuzzy number can be defined as an interval on real line as follows-

$$\tilde{A}^{\alpha} = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\} = \left[{}^{L}A^{\alpha},\ {}^{R}A^{\alpha}\right],$$

$$where\ {}^{L}A^{\alpha},\ {}^{R}A^{\alpha}\ are\ real\ numbers$$

Def. 2.2: LR -Type Fuzzy Numbers:

A generalized Fuzzy Number Ã = (a, b; β , γ) is said to be LR type if its membership function has two reference functions, known as shape function L & R such that

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a-x}{\beta})\ or\ f(x) & \text{for } x \in [a-\beta,\ a] \\ 1 & \text{for } x \in [a,b] \quad where\ \beta \geq 0, \gamma \geq 0 \\ R(\frac{x-b}{\gamma})\ \text{or } g(x) & \text{for } x \in [b, b+\gamma] \end{cases}$$

If the left and right spread functions f(x) & g(x) are linear then the LR-type fuzzy number is said to be linear LR-type fuzzy number. Triangular and trapezoidal fuzzy numbers are examples of linear LR-type fuzzy numbers.

This LR-type fuzzy number can be shown as in Fig-1.

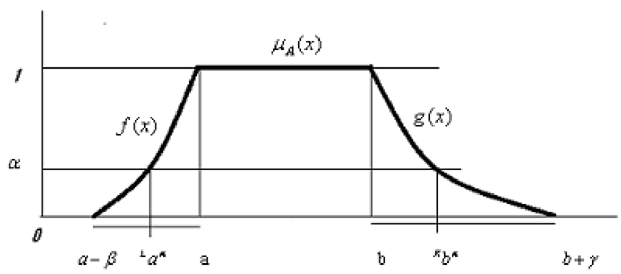

Fig 1. LR- type fuzzy Number $\tilde{A} = (a,b;\beta,\gamma)$

Def. 2.3: Generalized Ranking Value (GRV) of LR-Type Fuzzy Number:

The generalized ranking value was proposed by Wang-Yang [23] and that is the rectangular area between centroid of fuzzy numbers and the origin (0, 0). The centroid point of a fuzzy numbers, denoted by $(\bar{x},\bar{y})$, for a LR - type fuzzy numbers Ã it is defined as –

$$\bar{x}(\tilde{A}) = \frac{\int_{a-\beta}^{a} x.f(x)dx + \int_{a}^{b} x.dx + \int_{b}^{b+\gamma} x.g(x).dx}{\int_{a-\beta}^{a} f(x)dx + \int_{a}^{b} dx + \int_{b}^{b+\gamma} g(x).dx.}$$

$$\bar{y}(\tilde{A}) = \frac{\int_{0}^{1} y\{g^{-1}(y) - f^{-1}(y)\}.dy}{\int_{0}^{1} \{g^{-1}(y) - f^{-1}(y)\}.dy}$$

where $f(x)$, $g(x)$ are the left and right shape functions of fuzzy number Ã respectively and $f^{-1}(y)$ & $g^{-1}(y)$ are the inverse functions of $f(x)$ & $g(x)$ respectively.

The rectangular area between the centroid point and the origin point of the fuzzy number Ã is defined as-
area(Ã) = $\bar{x}.\bar{y}$

Therefore, the Generalized Ranking Value will be a crisp value $\bar{x}.\bar{y}$ and is used to compare the fuzzy numbers i.e.
GRV(Ã) = $\bar{x}.\bar{y}$ = area **(**Ã)

Suppose $\tilde{A}$ & $\tilde{B}$ are two LR-type fuzzy numbers then

$$\tilde{A} \geq \tilde{B} \;\; iff \;\;\; GRV(\tilde{A}) \geq GRV(\tilde{B})$$

and the value of $\bar{x}$ is known as ***Defuzzified Function Value*** [DFV] or mean value of the fuzzy number Ã.

Campbell, Dudek & Smith (CDS) Algorithm [6]:

According to CDS algorithm we can create a series of *(k-1)* auxiliary n-job 2 machine problems from the existing *nxk* job shop scheduling problem and then we proceed for Johnson algorithm for each of these *(k-1)* auxiliary problems to find the optimal sequence of jobs as well as make span. These *(k-1)* auxiliary series are generated using the following logics.

Step 1: For each r_{th} auxiliary problem, calculate the pseudo-machine processing time, where $r = 1, 2, ..., k-1$. whenever k-machines are in the systems. Processing time for the resulted 2-pseudo machine can be defined as $P_{i1}^{r} = \sum_{r} p_{ij}$, $P_{i2}^{r} = \sum_{k} p_{ij}$,
where p_{ij} is processing time of job *i* on the machine *j(1, 2, ..k)* & P_{ij} is the processing time of job *i* on the pseudo machine *j* *(j = 1,2)*

Step 2: By the procedure Step-1 we have created *(k -1)* auxiliary *nx2* job shop problem. Now by the help of Johnson Algorithm we determine the optimal sequence as well as make span time for each of the *(k -1)* auxiliary problems.

Step 3: Compare the make span time of the (*k -1)* sequences and select the optimal sequence i.e. the sequence with minimum make-span time.

The above CDS - algorithm has been established on the deterministic space with crisp processing time, its fuzzy version can be redefined for the fuzzy processing time of the pseudo machines as follows.

$\tilde{P}_{i1}^{r} = \sum_{j=1}^{r} \tilde{p}_{ij}$, $\tilde{P}_{i2}^{r} = \sum_{j=k+1-r}^{k} \tilde{p}_{ij}$

where $\tilde{p}_{ij}$ is the fuzzy processing time for ι^{th} job to j^{th} machine and $\tilde{P}_{i1}^{r}$ *and* $\tilde{P}_{i2}^{r}$ are fuzzy processing time for the r^{th} auxiliary sequence on pseudo machine 1 and 2 respectively where summation is fuzzy summation.

3. Problem Statement

Now we have to construct a model to optimize processing cost as well as make span time for the *n* x *k* job shop scheduling problem with non-equi-potential service station and the job processing time on k^{th} machine is given in the form of LR-type fuzzy number.

<u>The problem can be formulate as</u> :-

Table 3.1

Jobs	Work Station/parallel machine M_{11} M_{12} …. M_{1m}	Max processing time on M_1 p_i	Max processing time on machine M_i q_{ij} M_2 M_3 …M_k	No. of sub-jobs of type i
1	C_{11} C_{21} …. C_{m1} p_{11} p_{21} …… p_{m1}	p_1	q_{21} q_{31} …..q_{k1} r_{21} r_{31} …. r_{k1}	S_1
2	C_{12} C_{22} ….. p_{12} p_{22} ….. . …	p_2	q_{22} q_{32} ……q_{k2} r_{22} r_{32} ……r_{k2}	S_2
. .	.	..		..
n	C_{1n} …… C_{mn} p_{1n} …… p_{mn}	p_n	q_{2n} q_{3n} ……q_{nk} r_{2n} r_{3n} ……r_{nk}	S_n
	$\tilde{B}_1$ $\tilde{B}_2$ $\tilde{B}_m$ Processing time capacity of the work station			

Suppose there are m - non-equi-potential parallel machines $M_{11}, M_{12}, \ldots M_{1m}$ of type M_1 and the other machine is designate as M_j (where *j = 2, 3,…k*). Suppose n - jobs are to be complete in order $M_1M_2 \ldots M_k$ with no passing allowed. Also the jobs are assumed to be completed in the parts on the machine $M_{11}, M_{12}, \ldots M_{1m}$ and then jobs go through $M_2M_3..M_k$ for processing after completion on one or more then one on the machine M_{ij}. It is also given that machines $M_{11}, M_{12}, \ldots M_{1m}$ are available at time zero. Let $p_1, p_2\ p_3 \ldots p_n$ be the fuzzy processing time of jobs *1, 2, . . . n* on machine M_1 ; q_{ij} be the fuzzy processing time for i^{th} job on the machine M_j *(j = 2, 3,…k)*. $S_1, S_2, S_3, \ldots S_n$ are the no. of available sub-jobs of type $J_1, J_2, . . . J_n$ respectively . Let $\tilde{B}_1$, $\tilde{B}_2$, ... $\tilde{B}_m$ are fuzzy processing capacity (in form of time) of the machine $M_{11}, M_{12}, . . . M_{1m}$ respectively which is in the form of LR-type fuzzy numbers. C_{ij} denotes the operating cost per unit of i^{th} type job on machine M_{ij} for all $i = 1, 2, . . n ; j = 1, 2, . . m$ and p_{ij} is the processing time for per unit job i to machine M_{ij} $(J=1, 2, . . ., m)$. Now the problem is to find out minimum total elapsed time, allocation optimal manner and the total minimized processing cost. r_{ij} is the operating cost of j^{th} type of sub-job job on i^{th} machine $(i= 2,3,\dots,k; j= 1,2,\dots,n)$.

3.1 Assumptions

1) The jobs can be done in parts i.e. either a job can be completed wholly on one machine of type M_1 or on two or more than two up to m parallel service station of type M_1.
2) It is not necessary for a job to visit the entire non-equi-potential parallel machine.
3) The machine $M_{11}, M_{12}, . . . , M_{1m}$ are available at the same moment.
4) A job may not proceed to machine M_2 before it is finished on a machine of type M_1.
5) Each machine follows the same sequence of operations.
6) A jobs completed in the order $M_1, M_2, M_3 ... M_k$ with no passing allowed.

Now the problem is divided into two parts the Part-1 deals with the optimization of processing cost and the second part deals to find out optimal schedule and minimize completion time for all of the jobs.

3.1.1 Part One

The fuzzy integer generalized transportation problem can be formulated mathematically in the form of mathematical linear programming as follows. Let x_{ij} is the allocation of sub-jobs at i^{th} job in M_{ij} machine.

$$Minimize \sum_{i=1}^{n}\sum_{j=1}^{m} C_{ij}\ X_{ij} + \sum_{i=2}^{k}\sum_{j=1}^{n} r_{ij}\cdot S_{ij}$$

subject to

$$\sum_{j=1}^{m} r_{ij}\ x_{ij} \equiv S_i$$

$$\sum_{i=1}^{n} p_{ij}\ x_{ij} \equiv \tilde{B}_j$$

$$\sum_{j=1}^{m} p_{ij}\ x_{ij} \le \tilde{p}_i \ where\ x_{ij} is\ a\ positive integer$$

$$\tilde{B}_i = (\underline{a}_i, \bar{a}_i; \beta_{B_i}, \gamma_{B_i})_{LR}$$

$$\&\ \tilde{p}_j = (\underline{b}_j, \bar{b}_j; \beta_{p_j}, \gamma_{p_j})_{LR}\ are\ LR-type\ fuzzy\ number$$

$$for\ i\ \&\ j\ such\ that$$

$$i = 1,2,3,\dots,n\ ;\ j = 1,2,3\dots m;\ and$$

Since $\sum_{i=2}^{k}\sum_{j=1}^{n} r_{ij} S_j$ is known and definite real number, therefore our problem is only to find out the optimum solution of the following problem and this quantity added to the optimum value of objective function of the following problem.

$$\text{Minimize} \sum_{i=1}^{n}\sum_{j=1}^{m} C_{ij}\ X_{ij}$$

subject to

$$\begin{aligned} \sum_{j=1}^{m} x_{ij} &\equiv S_j \\ \sum_{i=1}^{n} p_{ij}\cdot x_{ij} &\le \tilde{B}_j \\ \sum_{j=1}^{m} p_{ij}\cdot x_{ij} &\le \tilde{p}_i \end{aligned} \qquad \dots(1)$$

$$x_{ij} \ge 0,\ \&\ is\ integer\ for\ i\ \&\ j\ such\ that$$

$$i = 1,2,3,\dots,n\ ;\ j = 1,2,3\dots m;$$

Let X is the solution matrix and its elements are corresponding decision variable i.e. $X = [x_{ij}]_{mn}$.

The above formulation of the objective means that the goal is also expressed by a fuzzy number. This fuzzy number is denoted by G and is expressed as
$G = (-\infty, c_0, 0, \beta_G)$

The following definition makes it clear how the satisfaction of the fuzzy constraints and of the fuzzy goal is understood in the problem (1).

Def. 1 Let X be an arbitrary solution of the problem (1) then
1. The value

$$\mu_C(x) = \min_{\substack{i=1,2,..m \\ j=1,2,...n}} \{\mu_{B_i}.(\textstyle\sum_{j=1}^{n} x_{ij}), \mu_{p_i}.(\sum_{i=1}^{m} x_{ij}.p_j)\}$$

is called the degree of satisfaction of the constraints of the problem (1) through X.

2. The membership value of the objective goal is

$$\mu_G(x) = \mu_G(C(x)) = \mu_{G(\sum_{j=1}^{n}\sum_{i=1}^{m} C_{ij} x_{ij})}$$

it is called the degree of satisfaction of the goal of problem (1) through X. According to Bellman Zadeh [25] approach, the optimal solution of the problem (1) is such a solution which satisfies the constraints as well as goal to a maximum degree.

Def 2: The optimal solution of the problem is X such that $\mu_D(x) = \min\{\mu_C(x),\ \mu_G(x)\}$, attains the maximum value for all x belongs to X. If maximum value is zero then we can say that problem has infeasible solution for that X. On the line of S. Chanas & D. Kuchta [7] the solution of the problem can be found by as the following process. As per Def-2, the problem has the equivalent solution, as to solving the following integer mathematical programming problem

$$Max(\min\{\mu_C(x),\ \mu_G(x)\})$$

$$s.t\ \ x_{ij} \ge 0\ \ and\ \ integer$$

$$for\ \ all\ \ i = 1,2,..m;\ \ j = 1,2,..n$$

The solution of the above problem is equivalent to solving the following one

$$Max(\alpha)$$

$$\begin{aligned} &subject\ to\ \ \mu_G(C(x)) \ge \alpha \\ &\mu_{B_i}\left(\sum_{j=1}^{n} x_{ij}\right) \ge \alpha \\ &\mu_{p_j}\left(\sum_{i=1}^{m} x_{ij}.p_j\right) \ge \alpha \end{aligned} \qquad \dots 2$$

$$1 \ge \alpha \ge 0,\ \ x_{ij} \ge 0\ \&\ \text{integer}\ \ for\ all\ \ i = 1...m;\ j = 1,...n.$$

$$\left(\sum_{i=1}^{m} x_{ij}.p_j\right) \le [{}^R p_j^\alpha] \ , \quad \left(\sum_{i=1}^{m} x_{ij}.p_j\right) \ge [{}^L p_j^\alpha]$$

$$\text{for any fix } \ \alpha, \ x_{ij} \ge 0 \,\&\, \text{integer } i = 1...m; j = 1,...n$$

$$[B_i^\alpha] \text{ is the maximum integer interval such that } [{}^L B_i^\alpha, {}^R B_i^\alpha] \subset B_i^\alpha$$

$$[p_j^\alpha] \text{ is the maximum integer interval such that } [{}^L p_j^\alpha, {}^R p_j^\alpha] \subset p_j^\alpha$$

$${}^L B_i^\alpha, {}^R B_i^\alpha, {}^L p_j^\alpha, {}^R p_j^\alpha \ \text{are all integer.}$$

Let B_i^α & p_j^α are the α–cut of LR-type fuzzy number respectively and are defined as follows

$$B_i^\alpha = \left[\underline{a}_i - L_i^{-1}(\alpha)\beta_{B_i} \ , \ \overline{a}_i + R_i^{-1}(\alpha)\gamma_{B_i}\right] \ , i = 1,2,..m$$

$$p_j^\alpha = \left[\underline{b}_j - L_j^{-1}(\alpha)\beta_{p_j} , \ \overline{b}_j + R_j^{-1}(\alpha)\gamma_{p_j}\right] \ , j = 1,2,..n$$

And the α –cut of the fuzzy goal G is the set

$$G^\alpha = (-\infty, c_0 + R_G^{-1}(\alpha).\gamma_G]$$

Now the problem 2 can be reformulate as follows

$$\begin{aligned}
&Max(\alpha)\\
&subject\,to\\
&C(x) \in G^\alpha\\
&\sum_{i=1}^{m} x_{ij} = S_j \ , \quad \left(\sum_{j=1}^{n} x_{ij}\right) \in B_i^\alpha \ , \quad \left(\sum_{i=1}^{m} x_{ij}.p_j\right) \in p_j^\alpha \qquad .....3\\
&1 \ge \alpha \ge 0, \ x_{ij} \ge 0 \,\&\, \text{integer for all } \ i = 1..m; j = 1,..n.\\
&B_i^\alpha \ , \ p_j^\alpha \ \text{are interval on real line.}
\end{aligned}$$

The above problem is an interval integer linear programming problem or interval transportation problem. This problem can be solved by any classical transportation programming problem algorithm, after transition from interval to classical one.

The solution of problem 3 or the max value of α for which the constraints of 3 are satisfied will be a real value α (the procedure to find out value of α is given separately on last). As we have obtained the maximum value of α, then the problem 1 can be converted into an interval linear programming problem which can be written as

$$\begin{aligned}
&Min \ \sum_{i}^{m}\sum_{j}^{n} c_{ij}.x_{ij}\\
&subject\,to\\
&\sum_{i=1}^{m} x_{ij} = S_j\\
&\left(\sum_{j=1}^{n} x_{ij}\right) \in [B_i^\alpha]\\
&\left(\sum_{i=1}^{m} x_{ij}.p_j\right) \in [p_j^\alpha]\\
&\text{for any fix } \ \alpha, \ x_{ij} \text{ is positive integer}\\
&[B_i^\alpha] \text{ is the maximum integer interval such that} -\\
&[{}^L B_i^\alpha, {}^R B_i^\alpha] \subset B_i^\alpha\\
&[p_j^\alpha] \text{ is the maximum integer interval such that} \qquad ...4\\
&[{}^L p_j^\alpha, {}^R p_j^\alpha] \subset p_j^\alpha\\
&\text{and } \ {}^L B_i^\alpha, {}^R B_i^\alpha, {}^L p_j^\alpha, {}^R p_j^\alpha \ \text{are all integers}\\
&\text{for } i = 1..m; \ j = 1,..n
\end{aligned}$$

The above problem 4 can be written in the classical linear programming form by restructured the constraints as follows

$$\begin{aligned}
&Min \ \sum_{i}^{m}\sum_{j}^{n} c_{ij}.x_{ij}\\
&subject\,to \qquad ...5\\
&\sum_{i=1}^{m} x_{ij} = S_j\\
&\left(\sum_{j=1}^{n} x_{ij}\right) \le [{}^R B_i^\alpha] \quad , \quad \left(\sum_{j=1}^{n} x_{ij}\right) \ge [{}^L B_i^\alpha]\\
&\left(\sum_{i=1}^{m} x_{ij}.p_j\right) \le [{}^R p_j^\alpha] \ , \quad \left(\sum_{i=1}^{m} x_{ij}.p_j\right) \ge [{}^L p_j^\alpha]
\end{aligned}$$

Problem 5 is the classical integer linear programming problem which can easily be solved by using LINGO optimization software for optimality and find out the optimal processing cost then added $\sum_{i=2}^{k}\sum_{j=1}^{n} r_{ij} \, .S_{ij}$ and the resulted cost is known as the total processing optimal cost of the given problem.

Algorithm to find out maximum value of α:

Step 1: *Set α(1)=0, α(2)=1;*
Step 2: *Solve problem 5 for α= α(2). If the problem is feasible and c(x(α(1)))∈$G^{\alpha(1)}$ then go to Step-3, otherwise stop problem has infeasible (μ_D(x)=0 for all x).*
Step3: *Solve problem-5 for α=α(2), if problem has feasible solution and c(x(α(2)))∈$G^{\alpha(2)}$, then stop x(α(2)) is the optimal solution of the problem-1 and μ_D(x(α(2)))=1, otherwise go to Step-4*
Step 4: *Set α(3)= (α(1)+α(2))/2 and go to Step-5.*
Step 5: *If problem has infeasible solution for α(3) then set α(2)= α(3) and go to step-6, otherwise there is possibilities of three cases that are*
1. μ_G(x(α(3)))= μ_c(x(α(3))) then x(α(3)) is an optimal solution of the problem-1 and Stop.
2. μ_G(x(α(3))) > μ_c(x(α(3))) then set α(1)= μ_c(x(α(3))) and go to Step-6.
3. μ_G(x(α(3))) < μ_c(x(α(3))) then set α(2)= μ_c(x(α(3))) or if α(2)= μ_c(x(α(3))) then α(2)= α(3) and go to Step-6.
Step 6: *If α(2)-α(1) > ε , then go to Step-4, otherwise check weather problem 5 for α= α(1) is the minimal extension of problem-5 for α= α(2). If not then go to Step-4, otherwise stop one of the solutions x(α(1)) or x(α(2)) is the optimal solution of problem-1. If the problem-3 was infeasible for α= α(2), then x(α(1)) is an optimal solution of problem-1.The number ε is decided by the user it can be as small as possible can be up to 0.05.*

3.1.2Part two

In the second part we will find the optimal job sequence and completion time for the reduced n x k job shop problem on the line of Johnson [11] and Kumar & other[13] algorithms. We have considered the exact (non-fuzzy) processing time on the parallel machines therefore and we fuzzify that time, taking left and right spread zero. Now we will find the solution of the above problem as follows

Assumptions & Notations

The following assumptions & notations have been used in the present chapter

Assumptions-

- Jobs are not preemptive.

- Each job consists of two tasks to be executed in sequence on two machines.
- The execution or processing time is given in the form of finite normalized fuzzy LR-type membership function for each job and is known.
- Each process on one machine started must perform till completion.
- A machine can process one job at a time.

Notations:-

$\tilde{A}$ Fuzzy Number A.

$\mu_{\tilde{A}}(x)$ *Membership function value at a point x in $\tilde{A}$.*

J Set of jobs to be processed ($J_1, J_2 .. J_n$)

n Number of Jobs in J.

$M_1, M_2, \ldots M_k$ = Machine on which jobs have to processed

${}^rM_j$ j^{th} pseudo machine for the r^{th} auxiliary problem, r= 1,2,...,(k-1).

$\tilde{p}_{ij}$ *Fuzzy processing time for the ι^{th} job on machine j (1, 2)*

i.e.

$\tilde{C}_{ij}$ *Fuzzy completion time for the i^{th} job at machine j.*

$\oplus$ *Fuzzy addition*

$\approx$ *Fuzzy Subtraction*

$\overset{n}{\underset{r=1}{(+)}}$ *Fuzzy Summation*

$\tilde{\max}$ *Fuzzy Maximization*

U Optimal sequence obtained by Johnson Algorithm ($T_1, T_2, \ldots, T_n$) such that for each T_i there exists J_j such that $T_i = T_j$ for all i, j, ∈ (1, 2, ..., n) i.e. ($T_1, T_2, T_3 \ldots T_n$) ≅ ($J_1, J_2, \ldots J_n$).

$\tilde{C}$ *the fuzzy completion time for all the jobs in the system.*

4. Proposed Algorithm

A generalized fuzzy Johnson algorithm for *n* x *k* job shop scheduling problem as follows

INPUT: - A set of *n*-jobs, each has to be processed on *k* -machines, each task has an LR-type fuzzy processing time membership function.

OUTPUT: -A fuzzy schedule with minimum fuzzy completion of *n*-job on each machine.

Step1:*Creates (k-1) auxiliary n-job 2- pseudo machine problems.*

Step2:*For each of (k - 1) auxiliary n x 2 problem finds GRV for each task with fuzzy execution time.*

Step3:*For each of (k -1) auxiliary n x 2 problem find optimal sequence U^r (r = 1, 2 ... k -1) and evaluate completion time of the original n x k problem for this sequence by using Johnson algorithm as follows*

$$\tilde{C}_{11} = \tilde{p}_{11},$$
$$\tilde{C}_{12} = \tilde{p}_{11} + \tilde{p}_{12}, \tilde{C}_{1j} = \tilde{C}_{1,j-1} + \tilde{p}_{1j}$$
$$\tilde{C}_{i1} = \tilde{C}_{i-1,1} + \tilde{p}_{i1},$$
$$\tilde{C}_{ij} = \tilde{\max}\{\tilde{C}_{i-1,j}, \tilde{C}_{i,j-1}\} + \tilde{p}_{ij},$$

where $\tilde{C}_{ij}$ is the fuzzy comletion time of i^{th} job on j^{th} machine.

Step4:*Select the optimal sequence out of (k -1) optimism sequence U^1, U^2 ... U^{k-1} with minimum GRV as well as minimum Defuzzified Function Value (DFV).*

Step5:*Set the final completion time of the sequence obtained from step 4 with fuzzy processing time on the line of Johnson procedure and use GRV to obtain fuzzy longer time.*

After step 5 we have obtained an optimal schedule with completion time in the form of fuzzy membership function.

Now we illustrate algorithm with the help of an example to describe the whole situation.

5. Example

Suppose *n*-jobs 2-machines job shop problem in which, machine M_1 has 2-equal non-equi-potential machines for same purpose. Processing time and processing cost of each job on each machine given in the form of crisp number. The time capacity of each machine for a particular sub job is given in the form of LR-type fuzzy number also processing time for each job of the machine is given in the form of fuzzy number. The problem is as follows.

Table 4.1

Jobs	Work Station/ Parallel Machine(cost, time per unit) M_{11} M_{12}	Processing Time p_i on M_1	Processing Time & cost q_i & r_{2i} on M_2	No. of Sub jobs S_i
J_1	10; 1.0 20; 0.8	(10,10;5,5)	(12,14;4,4);15	12
J_2	20; 0.9 50; 1.2	(9,9;4,4)	(9,10;2,2); 22	9
J_3	30; 1.1 60; 0.8	(1,1;1,1)	(10,11;5,5); 31	1
	(10,10;5,5) (16,16;5,5) Processing time capacity of parallel machines			

The problem can reduce in the linear programming form as follows-

Min $10.x_{11}+20.x_{21}+20.x_{12}+50.x_{22}+30.x_{13}+60.x_{23}+12*15+9*22+1*31$

Subject to.

$x_{11}+ x_{12} =12$, $x_{21}+ x_{22} = 9$, $x_{31} +x_{32} = 1$,

$x_{11}+x_{21}+x_{31} \equiv (10,10;5,5)_{LR}$

$x_{12}+x_{22}+x_{32} \equiv (16,16;5,5)_{LR}$

$x_{11}+x_{12} \equiv (10,10;5,5)_{LR}$

$x_{21}+x_{22} \equiv (9,9;4,4)_{LR}$

$x_{31}+x_{32} \equiv (1,1;1,1)_{LR}$

$x_{ij} \geq 0$ and integer i=1,2....m.; j=1,2,...n.

The fuzzy goal is determined by the following fuzzy number

$G=(0,300,0,500)_{LR}$

In the given example we are considering the case of power (square) function for left and right spread of fuzzy number.

The α–cut for the fuzzy numbers B_i & p_j are as follows

$B_1(\alpha) = [10-5\sqrt{(1-\alpha)}, 10+5\sqrt{(1-\alpha)}]$,

$B_2(\alpha) = [\, 16\text{-}5\sqrt{(1-\alpha)},\ 16+5\sqrt{(1-\alpha)}]$,
$P_1(\alpha) = [10\text{-}5.\sqrt{(1-\alpha)}, 10+5.\sqrt{(1-\alpha)}]$,
$P_2(\alpha) = [\, 9\text{-}4\sqrt{(1-\alpha)},\ 9+4\sqrt{(1-\alpha)}]$,
$P_3(\alpha) = [1\text{-}1.\sqrt{(1-\alpha)}, 1+1.\sqrt{(1-\alpha)}]$,
$G(\alpha) = [0, 300+500.\sqrt{(1-\alpha)}]$

The step of the algorithm is as follows

Step 1: $\alpha(1)=0$, $\alpha(2)=1$;

Step2:Problem has feasible for $\alpha(1)=0$ *and* $c(x(0))=150) \in G^0=[0,800]$.

Step 3: Problem has feasible solution for $\alpha=1$.

Step 4: $\alpha(3)=0.5$.

Step5: Problem has feasible solution for $\alpha(3)=0.5$ *and* $\mu_G(x(\alpha(3)))=0.74 > \mu_c(x(\alpha(3)))=0.5$, *therefore* $\alpha(1)=0.5$.

Step 6: $\alpha(2) - \alpha(1)=1\text{-}0.5=0.5>0.05$.

Step 4: $\alpha(3)=(0.5+1)/2=0.75$.

Step 5: Problem has infeasible solution for $\alpha=0.75$, *therefore* $\alpha(2)=0.75$.

Step 6: $\alpha(2) - \alpha(1)=0.75\text{-}0.5=0.25>0.05$.

Step 4: $\alpha(3)=(0.75+0.5)/2=0.625$.

Step 5: Problem has feasible solution for $\alpha=0.625$ *and* $\mu_G(x(0.625))=0.54 > \mu_c(x(0.625)))=0.64$, *therefore* $\alpha(2)=0.64$.

Step 6: $\alpha(2) - \alpha(1)=0.6\text{-}0.5=0.1>0.05$.

Step 4: $\alpha(3)=(0.5+0.6)/2=0.55$.

Step 5: Problem has feasible solution for $\alpha(3)=0.55$ *and* $\mu_G(x(\alpha(3)))=0.58 > \mu_c(x(\alpha(3)))=0.6$, *as* $\alpha(2)=\mu_c(x(\alpha(3)))=0.6$ *therefore* $\alpha(2)=0.55$.

Step 6: $\alpha(2) - \alpha(1)=0.55\text{-}0.5\leq 0.05$ *but problem for* $\alpha = \alpha(1)=0.5$ *is not a minimal extension of the problem for* $\alpha = \alpha(2)=0.55$ *and step-4 is performed once more.*

Step 4: $\alpha(3)=(0.5+0.55)/2=0.525$.

Step 5: Problem has feasible solution for $\alpha(3)=0.525$ *and* $\mu_G(x(\alpha(3)))=0.7 > \mu_c(x(\alpha(3)))=0.549$, *therefore* $\alpha(1)=0..549$.

Step 6: $\alpha(2) - \alpha(1)=0.55\text{-}0.549\leq 0.05$. *There fore it is checked whether problem 5 for* $\alpha = \alpha(1)=0.549$ *is a minimal extension of problem 5 for* $\alpha = \alpha(1)=0.55$. *Since the answer is positive hence it's the end of our algorithm, and one of the solution* $x(.549)$ *and* $x(0.55)$ *is the optimal solution of the problem 1.*

The optimal cost is = 510

And the maximum value of α for which problem has feasible solution is $\alpha= 0.55$.Hence for $\alpha= 0.55$

$B_1(0.55) = [10\text{-}5\sqrt{(1-0.55)}, 10+5\sqrt{(1-0.55)}]=[6.64, 13.37]$,
$B_2(0.55) = [\, 16\text{-}5\sqrt{(1-0.55)},\ 16+5\sqrt{(1-0.55)}]=[12.64, 19.36]$
$P_1(0.55)=[10\text{-}5.\sqrt{(1-0.55)}, 10+5.\sqrt{(1-0.55)}]=[6.635, 13.365]$
$P_2(0.55) = [\, 9\text{-}4\sqrt{(1-0.55)},\ 9+4\sqrt{(1-0.55)}]=[6.31, 11.69]$
$P_3(0.55) = [1\text{-}1.\sqrt{(1-0.55)}, 1+1.\sqrt{(1-0.55)}]=[0.33, 1.67]$
$G(0.55) = [0, 300+500.\sqrt{(1-0.55)}]= [0, 636.5]$
$[B_1(0.55)]=[7, 13]$, $[B_2(0.55)]=[13, 19]$, $[P_1(0.55)]= [7, 13]$
$[P_2(0.55)] =[7, 11]$, $[P_3(0.55)] = [1, 1]$, $[G(0.55)]=[0, 636]$

Hence the problem reduce to equivalent classical integer linear programming problem as follows

Min $= 10.x_{11}+20.x_{21}+20.x_{12}+50.x_{22}+30.x_{13}+60.x_{23} +12*15+9*22+1*31$

Subject to.

$x_{11}+ x_{12} =12$, $x_{21}+ x_{22} = 9$, $x_{31} +x_{32} = 1$,
$x_{11}+x_{21}+x_{31} \leq 13$, $x_{11}+x_{21}+x_{31}\geq 7$, $x_{12}+x_{22}+x_{32}\leq 19$
$x_{12}+x_{22}+x_{32} \geq 12$, $x_{11}+x_{12}\leq 13$, $x_{11}+x_{12}\geq 6$, $x_{21}+x_{22} \leq 11$ $x_{21}+x_{22} \geq 7$, $x_{31}+x_{32} = 1$
$x_{ij} \geq 0$ *and integer* $i=1,2....m.;\ j=1,2,...n.$

Now solve the above problem with the help of LINGO optimization software to find an integer solution of the given problem and i.e. $x_{11}=0$, $x_{12}=12$, $x_{21}=8$, $x_{22}=1$, $x_{31}=0$, $x_{32}=1$;

the optimal processing cost of the given problem is
$= 510+15*12+22*9+31*1= 919$.

Now we have to find out the job schedule and Makespan time for the *3 x 2* problem

The above job scheduling problem can be written as

Table-4.2

Jobs	M_1	M_2
1	(10,10;5,5)	(12,14;4,4)
2	(9,9;4,4)	(9,10;2,2)
3	(1,1;1,1)	(10,11;5,5)

Exact time taken by parallel machines in place of $M_1$are *9.6, 8.4 and .08 for job 1, 2 & 3* respectively and its fuzzified processing time is given in table-4 and also of M_2.

Table-4.3

Jobs	M_1	M_1 fuzzified	GRV on M_1	M_2	GRV M_2
1	9.6	(9.6,9.6;0,0)	4.8	(12,14;4,4)	6.5
2	8.4	(8.4,8.4;0,0)	4.2	(9,10;2,2)	4.75
3	0.8	.(0.8,0.8;0,0)	0.4	(10,11;5,5)	5.25

Now by using Johnson algorithm the optimal schedule will be *3-2-1* and the Make span for the sequence *321* can be found as follows

Table-4.4

Jobs	M_1	M_2
3	(0.8,0.8;0,0)	(10.8,11.8;5,5)
2	(10.4,10.4;0,0)	(22.8,25.8;9,9)
1	(18.8,18.8;0,0)	(31.8,35.8;11,11)

Hence the Makespan time is ***(31.8, 35.8; 11, 11)*** with DFV ***33.40*** .

Therefore the solution of the given problem can be written as

- Total Processing cost = *919*.
- Allotment of sub jobs to parallel machines is $x_{11}=0$, $x_{12}=12$, $x_{21}=8$, $x_{22}=1$, $x_{31}=0$, $x_{32}=1$;
- Total completion time of all the jobs (sub jobs) is *(31.8, 35.8; 11, 11) with DFV 33.40.*
- The optimal schedule is 3-2-1

6. Remark

The above established model can be further extend and generalized after taking parallel machines in place of more than two workstations and also we can consider fuzzy processing time on the parallel machines.

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