Volcanic Activity as a Function of Planetary Size

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Abstract: ***The level of volcanism on a planet plays a significant role in shaping the surface and its characteristics. The purpose of this paper is to obtain a theoretical framework that relates the extent of volcanism on a terrestrial planet to its size, i.e., its radius. We use a simplified model of a planet which has been formed in a molten state and gradually cools to form a hard crust on the exterior. The underlying assumption in this model is that the molten liquid in the interior exists in hydrostatic equilibrium, and that there exists an isentropic temperature gradient. If we use the argument that volcanism is related to the hydrostatic pressure below the crust, basic mechanics and thermodynamics gives us the relation that we desire.***

Keywords: Volcanism, Isentropic Planet model, Crust formation, Density gradient, Pressure gradient

1. Introduction

It does not take careful scrutiny to realize that the surface of certain planets, like Venus, is radically different from that of certain others, like Mercury. The primary difference lies in the large number of active volcanoes on Venus that Mercury seems to lack quite visibly. This gives rise to certain questions:

- Why are certain planets fiery giants while others are docile and calm? In scientific terms, why are some planets more volcanic than others?
- Has volcanism got anything to do with the size of a planet?

A lot of factors can lead to preferential volcanism in one planet over another. But our aim was to approach the questions theoretically, using the basic mechanics of planet formation to reach some tangible results. We therefore introduce the idea of an 'ideal' planet, the properties of which make it simple to analyse yet retain an overall similarity to the dynamics of a real planet. The features which dictate the 'ideality' will be discussed shortly, but we should touch upon the basics of planet formation first.

A planet is typically formed out of accretion of celestial dust, which come together primarily due to attractive gravitational forces. Accretion of dust involves inelastic collisions, generating a lot of heat. This heat goes into melting the whole mass into a ball of magma that starts cooling off. Cooling takes place by all three heat transfer modes – conduction, convection and radiation.

Radiation is the primary mode of heat loss from the surface immediately after formation but as the planet cools, the upper crust forms. Beyond this point, cooling occurs through conduction through the solidified layer of crust. A density gradient is created within the planet, with the highest density of molten magma being at the core and the lowest density being at the point just below the crust.

The analysis in this paper deals with phenomena after the formation of a thin layer of crust.

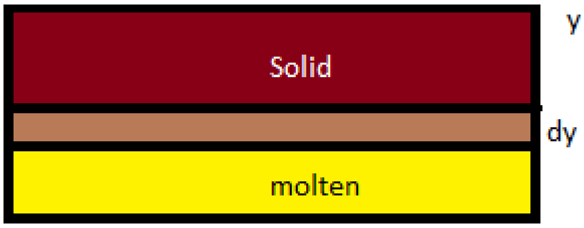
Volcanism is a consequence of the pressure of magma on the planet's crust. To quantify the degree of volcanic activity on a planet, it is therefore imperative that we obtain the pressure gradient inside a planet. It will depend upon its density gradient, as we shall see. But the density gradient inside is dependent on the radial distance from the centre of the planet. It changes mainly because of changes in pressure, temperature, crystalline nature and composition.

For purposes of simplicity, and considering our ideal model, crystalline nature and chemical composition are assumed to remain constant throughout the bulk of the planet.

2. Finding the Density Gradient and Pressure Gradient

2.1 Crust formation

Suppose, we have a tiny slice of the top surface of the planet which has cooled to form crust down to a depth y.



For an additional layer of thickness dy of crust to form in some small time dt, some dQ amount of heat has to leave this layer, and be conducted through the crust formed into space. From the principles of Calorimetry we have:

$$dQ = Ldm = L\rho A(dy)$$

dm is the mass of this layer, L is the latent heat of fusion of this material, A is its area, ρ is its density. Why we are considering the density to be a constant here will be explained at the end of the paper.

If a temperature gradient of ΔT exists between this layer and outside space, I may write the conduction equation to get (K is the thermal conductivity of the crust):

$$L\rho A(dy) = \frac{KA(\Delta T)(dt)}{y}$$

Integrating this differential equation for a total crust of thickness d to form in time t, we can show that:

$$\int_0^d y dy = \int_0^t c dt \quad \text{Or} \quad d^2 = 2ct$$

Here c is a constant including K, L and the temperature gradient ΔT. Of course assuming these to be constant requires explanation, and it shall be dealt with that shortly. But what is important about the solution is that we can conclude that:

$$d \propto \sqrt{t}$$

So given constant temperature gradient, and thermal conductivity (hence, in some way, uniform composition) the thickness of crust formed is proportional to the square root of the time taken to form the layer.

2.2 The criteria for being 'ideal'

Now, I finally present the assumptions for a planet to be ideal, and I hope that should justify most of the work till now. For ideality, the planet should satisfy the following:

1. Spherically symmetric, non-rotating terrestrial planet, not a gas giant
2. Homogeneous, i.e., has uniform composition
3. Molten matter in a state of hydrostatic equilibrium, no convection currents, implying no cooling by convective currents.
4. Density variation is only due to gravitational compression.
5. Temperature gradient inside is adiabatic

What it means to say that the temperature gradient inside the planet is adiabatic is the following [1]:

We imagine gravity absent. So there is no compression anywhere. Suddenly if we switch gravity on, the layers of dust cramp together, create pressure and heat. No heat is given or taken out of the system, because the heat released during accretion is used up to melt the planet into a ball of molten rock. But a temperature gradient is created inside the planet. Hence the process of its creation is **adiabatic**.

But an important conclusion from this is that the process being adiabatic, the entropy inside the planet has a constant value. Hence this model of the planet is **isentropic**.

2.3 Quantifying the isentropic planet model

Instead of taking the density a function of pressure and temperature, we take the density a function of pressure and the thermodynamic variable entropy (S). This is because thermodynamics allows the state of a system to be characterised by any two state variables. This is to exploit the fact that entropy remains constant in our ideal planet. Hence $dS = 0$.

Let *R* denote the radial distance from the centre of the planet. Using the chain rule from calculus, we can write:

$$\frac{d\rho}{dR} = \left(\frac{\partial \rho}{\partial P}\right)\left(\frac{dP}{dR}\right) + \left(\frac{\partial \rho}{\partial S}\right)\left(\frac{dS}{dR}\right)$$

Since $dS = 0$, we get: $\frac{d\rho}{dR} = \left(\frac{\partial \rho}{\partial P}\right)\left(\frac{dP}{dR}\right)$ (1)

Using the assumption that the entire pressure inside the planet is because of the gravitational compressive force, we form the equations that gives us the pressure inside the planet as a function of radius, and hence the density inside the planet as a function of the radius.

$$\frac{dP}{dR} = -\frac{4\pi G\rho}{R^2}\int_0^R \rho r^2 dr \qquad (2)$$

$$\frac{d\rho}{dR} = -\frac{4\pi G\rho^2}{BR^2}\int_0^R \rho r^2 dr \qquad (3)$$

Here $B = \frac{\partial \rho}{\partial P}$ is the bulk modulus of the material of the planet (as per our assumption that the planet is homogeneous)

Equation (2) can be obtained in various ways by computing the radial gravitational field at any point (r, θ, φ) inside the spherical planet and taking density to vary only radially. And then we technically use the result in (1) to get (3). Notice the r used as the variable of integration so that we may calculate the total mass of the planet enclosed within a sphere of radius R. The integration is necessary since the density is not at all uniform, rather it depends on the radial distance from the centre.

Differentiating (3) with respect to R above and using (2), we get:

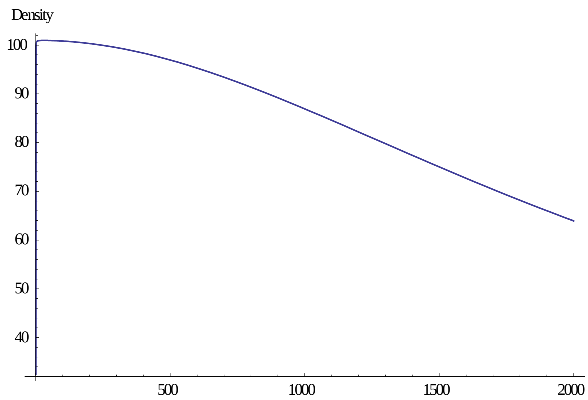
$$\frac{d^2\rho}{dR^2} - \frac{2}{\rho}\left(\frac{d\rho}{dR}\right)^2 + \frac{2}{R}\left(\frac{d\rho}{dR}\right) + \frac{4\pi G}{B}\rho^3 = 0$$

Here we have used the Leibnitz Theorem for differentiation under the integral sign:

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x)).h'(x) - f(g(x)).g'(x)$$

2.4 Solving the Differential Equations

The differential equation obtained above can be solved using any programming language. Wolfram Mathematica was used to plot the solution function with density on the y-axis and the planetary radius on the x-axis. The result obtained is shown below for some appropriate values of the parameters in the differential equation.



With a simple approximation, I can take the argument further:

We notice from the graph that there is almost a linear decrease of density with radius. So we can assume, for the propagation of our theory that: $\rho_0 - aR$

(Here the constant a is the slope containing the constants B, G etc. ρ_0 is the density at the centre of the planet)

Considering the above approximation, we solve the differential equation for pressure with suitable boundary conditions obtaining the following:

$$P = \frac{\pi G}{3}[2\rho_0{}^2(A^2 - R^2) - \frac{7}{3}\rho_0 a(A^3 - R^3) + \frac{3}{4}a^2(A^4 - R^4)]$$

Where A is the radius of the planet.

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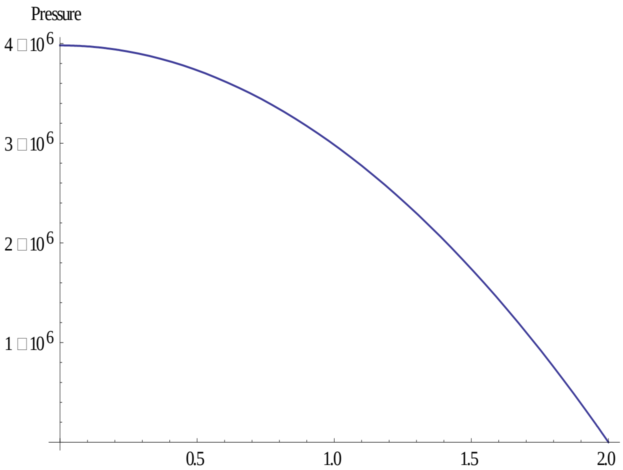
This is the graph showing pressure variation with the planetary radius for some appropriate parameters. It is as it should be, with the pressure being the highest at the centre (where R = 0) and falling to zero at the surface (where R=A).



The graph has been plotted again, on Wolfram Mathematica. The above approximate relation helps us to approximate the variation of volcanic activity with the radius of the planet.

3. Deductions and Results

3.1 The Critical Arguments:

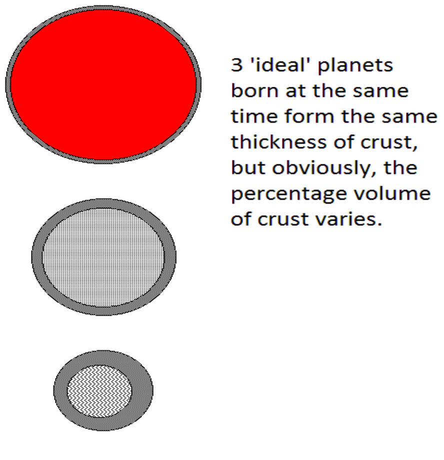
Nearing the end of my analysis, I only have a few critical arguments to present, which shall lead to my conclusion:

- Volcanism in a planet depends on the pressure under the crust.
- For two planets born at the same time, their crusts have cooled to the same thickness, say d, in fixed time T. Since $(d \propto \sqrt{t})$

We want the pressure just below the crust, i.e., the pressure at radial distance:

$$R = R_p - d = A - d$$

A is the radius of the planet in question.



- The density at the centre ρ_0, is greater for larger planets, due to greater depth from the surface (hence greater pressure)

When $R = A - d$:

$A^2 - R^2 = (2A - d)d = X$

$A^3 - R^3 = d(A^2 + d^2 + Ad) = Y$

$A^4 - R^4 = d(2A - d)(2A^2 + d^2 + 2Ad) = Z$

So with all these arguments and using the following approximate expression for pressure:

$$P = \frac{\pi G}{3}[2\rho_0{}^2(A^2 - R^2) - \frac{7}{3}\rho_0 a(A^3 - R^3) + \frac{3}{4}a^2(A^4 - R^4)]$$

When $R = A - d$, we write the pressure just under the crust P_c as:

$$P_c = \frac{\pi G}{3}[2\rho_0{}^2 X - \frac{7}{3}\rho_0 Y + \frac{3}{4}a^2 Z]$$

This quantity is obviously larger for larger planets, since all terms and variables contained within at are positive, and even ρ_0 is larger for larger planets. As A increases, ρ_0 increases and so does the pressure under the crust P_c.

3.2 Conclusion

Our analysis reveals that for planets of same homogeneous chemical composition, which have lived equal time t, the crust thickness is same, i.e., d. So the pressure just below the crust is proportional to A, the radius of the planet. Hence bigger planets shall have greater pressure below the crust and more chance for magma to break through weak spots and form a volcano. Thus a larger planet is more volcanic.

3.3 A note:

There might be a question that if density varies with radius, how come a constant ρ was used when I was talking about crust formation.

We have $\rho = \rho_0 - aR$.

When $R = A - d$,

$\rho = \rho_0 - a(A - d)$

Or $\rho = \rho_0 - aA\left(1 - \frac{d}{A}\right) = \rho_0 - aA \quad (because\ d \ll A)$

a and A being constants, density at around the surface is almost constant, and carrying the analysis forward, the same can be said of the temperature, so that we can assume a constant ΔT. The key point to note is that the thickness of the crust is very small as compared to the radius of the entire planet.

4. Achievements and Limitations

This paper deals with the comparative study of the degree of volcanism between two different planets formed at the same time with the assumption that the planet is 'ideal', i.e., has the properties as discussed above. However real planets are far from this, as there are different kinds of magmatic flows within the molten interior and so heat loss occurs by convection as well, which has been completely ignored in this paper for the sake of simplicity.

However, even a more rigorous study of the interior reveals a very similar pressure gradient. The density profile we obtained was a monotonically decreasing function of radius. Exact studies in the past have shown flat regions in the curve of the density profile, but on the whole, the density indeed decreases with increase in radius. Herein lies the success of our model – a lot was inferred from it, with hardly any complicated mathematics.

5. Acknowledgements

I am indebted to Dr Annu Malhotra, St Stephen's College, University of Delhi for giving me the opportunity to undertake this research all by myself. Dr Geetanjali Sethi, St Stephen's College, University of Delhi, was kind enough to offer me valuable advice when I needed it, particularly at the stage when the differential equations were to be obtained. My peers and family have been excellent motivators and it is to them, that I dedicate this work.

References

[1] Notes on Geology and the Adam-Williamson Equation from Cornell University, sourced from the website http://www.geo.cornell.edu/geology/classes/geol388/pdf_files/density2.pdf
[2] www.wikipedia.org for additional data on the Adam-Williamson equation

Author Profile

Mitrajyoti Ghosh, is currently a student of Physics at St Stephen's College, University of Delhi. He is about to start his 3rd year of college this July.