An Integral Equation Involving H- Function of Two Variables as Its Kernel

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Abstract: The object of this paper is to solve an integral equation of convolution form having the H- function of two variables as its kernel. It generalizes the result given by Buschman, Karl and Gupta [8, pp. 226-228]. A few other special cases are also given.

1. Introduction

The H- function of two variables defined by Mittal and Gupta [5, p.117] is represented here in contracted form as

\[ H[X,Y] = H^{0N;m,n;g,h}_{P,Q;p,q;u,v} \left[ x \left( a_p, \alpha_p, A_p \right) ; \left( c_p, C_p \right) ; \left( e_u, E_u \right) \right] \]

\[ y \left( b_q, \beta_q, B_q \right) ; \left( d_q, D_q \right) ; \left( f_v, F_v \right) \]

\[ = \frac{1}{(2\pi)^2} \int_{L_1} \int_{L_2} \phi_1(s) \phi_2(t) \psi(s,t) x^s y^t dsdt \] (1.1)

Where \( \psi(s,t) = \prod_{j=1}^{N} \Gamma(1-a_j + \alpha_j s + A_j t) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j s - A_j t) \) (1.2)

\[ \phi_1(s) = \prod_{j=1}^{m} \Gamma(d_j - D_j s) \prod_{j=1}^{n} \Gamma(1-c_j + C_j s) \prod_{j=m+1}^{q} \Gamma(1-d_j + D_j s) \prod_{j=n+1}^{p} \Gamma(c_j - C_j s) \] (1.3)

\[ \phi_2(t) = \prod_{j=1}^{g} \Gamma(f_j - F_j t) \prod_{j=1}^{h} \Gamma(1-e_j + E_j t) \prod_{j=g+1}^{v} \Gamma(1-f_j + F_j t) \prod_{j=h+1}^{u} \Gamma(e_j - E_j t) \] (1.4)

Where \( x, y \neq 0 \), an empty product is interpreted as unity.

\( N, P, Q, m, n, p, q, g, h, u, v \) are all non-negative integers such that

\[ 0 \leq N \leq P, Q \geq 0, 0 \leq m \leq q, o \leq n \leq p, 0 \leq g \leq v, 0 \leq h \leq u \) and \( \alpha_j, \beta_j, A_j, B_j, C_j, D_j, E_j, F_j \) are all positive. The sequence of the parameters \( (a_p), (b_q), (c_p), (d_q), (e_u), (f_v) \) are so restricted that none of the poles of the integrand coincide.

The double integrand in (1.2.1) converges absolutely if

\[ |\arg x| < \frac{1}{2} \Delta_1 \pi, \quad |\arg y| < \frac{1}{2} \Delta_2 \pi \] (1.5)
Where \( \Delta_1 = \sum_{j=1}^{N} (\alpha_j) - \sum_{j=N+1}^{P} (\alpha_j) - \sum_{j=1}^{Q} (\beta_j) + \sum_{j=1}^{m} (D_j) - \sum_{j=m+1}^{q} (D_j) + \sum_{j=1}^{n} (C_j) - \sum_{j=n+1}^{p} (C_j) > 0 \) \hspace{1cm} (1.6)

\[ \Delta_1 = \sum_{j=1}^{N} (A_j) - \sum_{j=N+1}^{P} (A_j) - \sum_{j=1}^{Q} (B_j) + \sum_{j=1}^{g} (F_j) - \sum_{j=g+1}^{v} (F_j) + \sum_{j=1}^{h} (E_j) - \sum_{j=h+1}^{u} (E_j) > 0 \] \hspace{1cm} (1.7)

2. Notations and Results Used

The Lplace transform

\[ F(p) = \int_{0}^{\infty} e^{-pt} f(t) dt, \quad \text{Re}(p) > 0 \]

is represented by \( F(p) = f(t) \) \hspace{1cm} (2.1)

**ERDELYI [2, p.129-131]**

If \( f(t) = F(p) \), \( f(0) = f'(0) = ....... f^{n-1}(0) = 0 \) and \( f^n(t) \) is continuous, then

\[ f^n(t) = p^n F(p) \] \hspace{1cm} (2.2)

If \( f_1(t) = F_1(p) \) and \( f_2(t) = F_2(p) \), then

\[ \int_{0}^{t} f_1(u) f_2(t-u) du = F_1(p)F_2(p) \] \hspace{1cm} (2.3)

**SRIVASTAVA [8, P.19.]**

\[ H^{1,p}_{p,q+1} \left[ -x \begin{pmatrix} 1-a_j, A_j \cr 0,1 \end{pmatrix} x \begin{pmatrix} b_j, B_j \cr 1-b_j, B_j \end{pmatrix} \right] = \psi_q \left[ \begin{pmatrix} (a_j, A_j) \cr (b_j, B_j) \end{pmatrix} x \right] = \sum_{r=0}^{P} \prod_{j=1}^{P} \Gamma(a_j + A_j r) x^r \prod_{j=1}^{Q} \Gamma(b_j + B_j r) r! \] \hspace{1cm} (2.4)

**NAIR [7, P.10]**

\[ t^a \psi_1 \left[ (d, 1); (1 + \alpha, b), -ct^b \right] = \Gamma(d) p^{-\alpha-\gamma} (1 + cp^{-b})^{-\gamma} \] \hspace{1cm} (2.5)

Provided \( \text{Re}(p) > 0, 2 > b > 0, \text{Re}(1 + \alpha) > 0, \left| \text{arg} cp^{-b} \right| < \frac{\pi(2-b)}{2} \).

**MUHAMMED [6, P.109]**

\[ \int_{0}^{\infty} x^{\alpha-1}(1-x)^{\beta-1}\psi_{\alpha}(m, n) \psi_{\alpha}(m, n); z(ax)^{\alpha}\psi_{\alpha}(m, n) \psi_{\alpha}(m, n); (ax+b)(1-x)^{\gamma} \] \[ \times H(z_1(ax)+b(1-x))^{\gamma} \psi_{\alpha}(m, n) \psi_{\alpha}(m, n); z_2(ax)^{\gamma} \psi_{\alpha}(m, n) \psi_{\alpha}(m, n); (ax+b)(1-x)^{\gamma} \] \[ dx \]
\[ a^{-\alpha} b^{-\beta} \sum_{d=0}^{\infty} \frac{\prod_{j=1}^{r} \Gamma(m_j + M_j d) z^d}{\prod_{j=1}^{s} \Gamma(n_j + N_j d) d!} \]

\[ H_{0,N+2(m,n),l,g}(p,q) \begin{bmatrix} z_1 & (1 - \alpha - d\rho, \lambda_1, \lambda_2), (1 - \beta - d\sigma; \mu_1, \mu_2), ((a_p, \alpha_p, A_p); (e_{b}, E_{b})) \end{bmatrix} \]

\[ H_{0,N+2(m,n),l,g}(p,q) \begin{bmatrix} z_2 & (b_2, \beta_2, B_2), (1 - \alpha - \delta - d\rho, \lambda_1 + \mu_1, \lambda_2 + \mu_2); ((d_q, D_q); (f_q, F_q)) \end{bmatrix} \]

(2.6)

provided

\[ \rho, \sigma, \lambda_1, \lambda_2, \mu_1, \mu_2 > 0 \quad \text{Re}(\alpha + \frac{\lambda_i d_j}{D_j} + \frac{\lambda_2 f_k}{F_k}) > 0 \quad \text{Re}(\beta + \frac{\mu_i d_j}{D_j} + \frac{\mu_2 f_k}{F_k}) > 0 \]

(j = 1, \ldots, m) and (k = 1, \ldots, g)

\[ 1 + \sum_{i} N_j - \sum_{j} M_j > 0 \quad |\arg z_1| < \frac{1}{2} \pi \Delta_1, \quad |\arg z_2| < \frac{1}{2} \pi \Delta_2, \quad \Delta_1, \Delta_2 > 0 \]

\[ \Delta_1, \Delta_2 \] is defined by (1.6) and (1.7) respectively.

\[ H\{x, y\} \] defined by (1.1) and \( ax + b(1 - x) \) is not zero for \( 0 \leq x \leq 1, \ a, b \neq 0 \)

Buschman, Kurl and Gupta [1]

\[ L(x-1)^{-\alpha \beta} H_{0,0,0,0,1,0,1,1}(p,q) \begin{bmatrix} x-t & (a_j, \alpha_j, A_j); (c_j, C_j); (e_j, E_j) \end{bmatrix} \begin{bmatrix} x-t & (b_j, \beta_j, B_j); (0,1), (d_j, D_j); (0,1), (f_j, F_j) \end{bmatrix} \]

\[ = p^{-\sigma} H_2[p^{-1}, p^{-1}] \]

Provided \( \text{Re}(\sigma + \frac{d_j}{D_j} + \frac{f_k}{F_k}) > 0 \),

Where \( H_2[p^{-1}, p^{-1}] = \)

\[ H_{0,1,0,1,0,1,1}(p,q) \begin{bmatrix} p^{-1} & (1 - \sigma, 1, 1), (a_j, \alpha_j, A_j); (c_j, C_j); (e_j, E_j) \end{bmatrix} \begin{bmatrix} p^{-1} & (b_j, \beta_j, B_j); (0,1), (d_j, D_j); (0,1), (f_j, F_j) \end{bmatrix} \]

\[ = \sum_{M,N=0}^{\infty} C_{M,N} (-p)^{-M-N} \Gamma(\sigma + M + N)! / M! N! \]

(2.8)

Where \( C_{M,N} = \phi(N,M)\theta_2^i(N)\theta_3^j(M) \)

\[ H_2[p^{-1}, p^{-1}] = \sum_{v=0}^{\infty} h_v p^{-v} \]

(2.10)

Where \( h_v = (-1)^v \Gamma(\sigma + v) / v! \sum_{\mu=0}^{v} v \left( \begin{array}{c} v \\ \mu \end{array} \right) C_{v-\mu, \mu} \)

(2.11)

If \( k \) denotes the least value of \( v \) for which \( h_v \neq 0 \), then

\[ H_2[p^{-1}, p^{-1}] = p^{-k} \sum_{n=0}^{\infty} h_{k+n} p^{-n} \]

(2.12)

So that if we let the coeifficients \( H_j \) be determined by the relation

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\[
\left[ \sum_{n=0}^{\infty} h_{k+n} p^{-n} \right]^{-1} = \sum_{\lambda=0}^{\infty} H_{\lambda} p^{-\lambda}
\]  
(2.13)

\[
t^{-p-k-\sigma-1} \sum_{\lambda=0}^{\infty} H_{\lambda} t^{\lambda} / \Gamma(p-k+\lambda-\sigma) = p^{-(p-k-\sigma)} \sum_{\lambda=0}^{\infty} H_{\lambda} p^{-\lambda}
\]  
(2.14)

The coefficient \( H_{\lambda} \) being defined by the recurrences

\[ h_k H_0 = 1 \quad \text{and for} \quad \mu > 0 \quad \text{by} \quad \sum_{\lambda=0}^{\mu} H_{\lambda} h_{\mu+k-\lambda} = 0 \quad \text{and the power series coefficients} \quad h_{\lambda} \quad \text{is given by} \quad (2.11). \]

3. The Integral Equation.

Theorem.

If \( 2 > b > 0, \quad \text{Re}(1+\alpha+a) > 0, \)

\[ A B \Gamma(d) \Gamma(-d) = 1, \quad \alpha + c + 2 = m + l, \quad f^{q}(0) = 0, \quad 0 \leq q < m; \]

\[ g^{s}(0) = 0, \quad 0 \leq s < h; \quad m, l, h \quad \text{are integers and} \quad \text{Re}(h-l-k+\alpha+c+1) > 0, \]

then the integral equation:

\[
g(x) = A \sum_{r=0}^{\infty} \frac{\Gamma(d + r)}{r!} \int_{0}^{x} (x-t)^{a+\alpha+br} H^{0,1}_{p_{1}+1,q_{1}+1; p_{2}, q_{2}+1; p_{3}, q_{3}+1} \left[ x-t \right] \left[ 1-(\alpha;1,1) (a_{j}, \alpha_{j}, A_{j}) ; (c_{j}, C_{j}) ; (e_{j}, E_{j}) \right] \left[ x-t \right] \left[ (b_{j}, \beta_{j}, B_{j}) , (-a-\alpha-br;1,1) ; (0,1), (d_{j}, D_{j}) ; (0,1), (f_{j}, F_{j}) \right] \left[ D^{m} f(t) dt \right] \]

(3.1)

Where the H- function of two variables involved in (3.1) satisfies all the conditions corresponding appropriately to the set of convergence conditions (1.5) to (1.7), the solution of the integral equation is given by

\[
f(t) = B \sum_{r=0}^{\infty} \frac{\Gamma(-d + r)}{r!} \int_{0}^{x} (t-x)^{h-l+c+br-k-\alpha} V(t-x) [D^{l} g(x)] dx
\]  
(3.2)

where

\[ V(x) = \sum_{\lambda=0}^{\infty} \frac{H_{\lambda} x^{\lambda}}{\Gamma(br + c + \lambda + h-k-\alpha)}
\]  
(3.3)

The coefficients \( H_{\lambda} \) being defined by the recurrences \( H_{\lambda} h_{0} = 1 \quad \text{and for} \quad \mu > 0 \quad \text{by} \quad \sum_{\lambda=0}^{\infty} H_{\lambda} h_{\mu+k-\lambda} = 0 \quad \text{and the power series coefficients} \quad h_{\lambda} \quad \text{given by} \quad (2.10). \)
PROOF:

Let \( f(t) \equiv F(p) \) and \( g(t) \equiv G(p) \) then

From (2.5), (2.7) and (2.11)

\[
\begin{align*}
\int_{0}^{\infty} \gamma_{1}^{(d,1), (a+1,b); zt^{b}} \Delta \gamma_{1}^{(d)} p^{-1-a} (1 - zp^{-b})^{-d} \\
\int_{0}^{\infty} \gamma_{1}^{(-d,1), (c+1,b); zt^{b}} \Delta \gamma_{1}^{(-d)} p^{-1-c} (1 - zp^{-b})^{-d}
\end{align*}
\]

(3.4)
(3.5)

\[
H_{0,0}^{0,1, n} \Delta \gamma_{1}^{(d,1), (a+1,b); zt^{b}} (x-t)^{-a-1}
\]

Using (2.3) in (3.4) and (3.6)

\[
\Gamma(d) p^{-1-a} (1 - zp^{-b})^{-d} \sum_{n=0}^{\infty} h_{k+n} p^{-n}
\]

(3.6)

In (3.7) put \( t = ux \) and evaluate using (2.6) to get:

\[
\Gamma(d) p^{-1-a-a-k} (1 - zp^{-b})^{-d} \sum_{n=0}^{\infty} h_{k+n} p^{-n} = \sum_{r=0}^{\infty} \frac{(d+r) z r}{r!} x^{a+alpha r} \Delta \gamma_{1}^{(d,1), (a+1,b); zt^{b}}
\]

(3.7)

(3.8)

Now using (2.3) in (3.5) and (3.9), to get:

\[
\begin{align*}
\int_{0}^{\infty} \gamma_{1}^{(d,1), (a+1,b); zt^{b}} \Delta \gamma_{1}^{(d)} p^{-1-a} (1 - zp^{-b})^{-d} \\
\int_{0}^{\infty} \gamma_{1}^{(-d,1), (c+1,b); zt^{b}} \Delta \gamma_{1}^{(-d)} p^{-1-c} (1 - zp^{-b})^{-d}
\end{align*}
\]

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\[
\Gamma(-d) p^{-1-c} (1 - zp^{-b} )^d \sum_{\lambda=0}^{\infty} H_\lambda p^{-\lambda} = \int_0^x t^{c-1} \psi([(-d,1),(c+1,b) ; zt^b] (x-t)^{h-1-k-\alpha-1} \sum_{\lambda=0}^{\infty} \frac{H_\lambda (x-t)^\lambda}{\Gamma(h-1-k+\lambda-\alpha)} dt
\]

In (3.10) put \( t = ux \) and evaluate using (2.14), to get:
\[
\Gamma(-d) p^{-1-c+k-k-h} (1 - zp^{-b} )^{-d} \sum_{\lambda=0}^{\infty} H_\lambda p^{-\lambda} = \sum_{r=0}^{\infty} \frac{\Gamma(-d+r)z^r}{r!} x^{h-1+c+b-r-\alpha} \sum_{\lambda=0}^{\infty} \frac{H_\lambda x^\lambda}{\Gamma(br+c+\lambda+h-k-\alpha)}
\]

Using (2.3) and (3.8), the integral equation (3.1) becomes
\[
G(p) = A\Gamma(d) p^{m-1-a-a-k} (1 - zp^{-b})^{-d} \sum_{n=0}^{\infty} H_{k+n} p^{-n} F(p)
\]

Similarly using (2.3) and (3.10), the integral equation (3.2) becomes
\[
F(p) = B\Gamma(-d) p^{1-c+a+k} (1 - zp^{-b})^{-d} \sum_{\lambda=0}^{\infty} H_\lambda p^{-\lambda} G(p)
\]

The equations (3.12) and (3.13) can be obtained from each other when
\[
AB\Gamma(-d)\Gamma(d) = 1 \quad \text{and} \quad m+1 = a+c+2.
\]

Hence by LERCH’S theorem [4,p.5], it follows that each of the integral equation (3.1) and (3.2) is the solution of the other.

4. Special Cases

In the theorem use [8,pp.88,89, eqn (6.4.3)] to get:

The integral equation
\[
g(x) = A \sum_{r=0}^{\infty} \frac{\Gamma(d+r)\Gamma(\alpha)}{\Gamma(a + \alpha + br + 1)} \int_0^x (x-t)^{a+\alpha+br} \times F(p_1+1;p_2;p_3; q_1+1;q_2;q_3)
\]

\[
\times [D^m f(t)] dt
\]

Where
\[
2 > b > 0, \quad \text{Re}(1+\alpha+a) > 0, \quad \text{Re}(a_j) > 0, (j = 1,\ldots,p_1)
\]

\[
\text{Re}(c_j) > 0, (j = 1,\ldots,p_2), \quad \text{Re}(e_j) > 0, (j = 1,\ldots,p_3), \quad \text{Re}(1-b_j) > 0, (j = 1,\ldots,q_1), \quad \text{Re}(1-d_j) > 0, (j = 1,\ldots,q_2)
\]
\[
\text{Re}(1 - f_j) > 0, \quad (j = 1, \ldots, q_3) \quad A B I \Gamma(d)\Gamma(-d) = 1, \quad a + c + 2 = m + 1 \quad f^q(0) = 0, 0 \leq q \leq m
\]

\[
gs(0) = 0, 0 \leq s \leq h, \quad m, l, h \text{ are integers and } \quad \text{Re}(h - l - k - \alpha + c + 1) > 0 \text{ has for its solution.}
\]

\[
f(t) = B \sum_{r=0}^{\infty} \left[ \frac{\Gamma(-d + r)\Gamma(\alpha)}{r!} \int_{0}^{t} (t - x)^{h-1+c+r-l-k-\alpha} W(t - x)[D^h g(x)]dx \right]
\]

Where

\[
W(x) = \sum_{\lambda=0}^{\infty} \left[ \frac{H_{\lambda}x^2}{\Gamma(c + \lambda + br + h + l - 1 - k - \alpha)} \right]
\]

Where \(H_{\lambda}\) being determined by (3.4) and \(h^q\) being given by (2.11) and the coefficients in (2.9) reducing to the form \(c_{\nu-\mu,\mu} = \theta_2^l(v - \mu)\theta_3^l(\mu)\).

In (3.1) and (3.2) put \(z = m = l = 0, A = B = 1, a = c = -1, \alpha = \sigma, h = \rho\) to get the result given by Buschman, Kurl and Gupta[8, pp.226-228]

In (3.14) put \(z = m = l = 0, A = B = 1, a = c = -1, \alpha = \sigma, h = \rho\)

\[
p_1 = q_1 = 0, (C_j) = (D_j) = (E_j) = (F_j) = 1, n_2 = n_3 = p_2 = p_3 = q_2 = q_3 = 1,
\]

\[
c_1 = 1 - a_1, d_1 = 1 - b_1, e_1 = 1 - a_2, f_1 = 1 - b_2\]

and use the result [8, p.161] to get another result given by Buschman, Kurl and Gupta[8, p.230].

References


