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Common Coupled Fixed Point Theorems for Geraghty - Type Contraction Mappings Using Monotone Property in Partially Ordered Partial b-Metric Spaces

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Abstract: ***In this paper, we prove common coupled fixed point theorems for Geraghty - type contraction mappings using monotone property in partially ordered partial b - metric spaces. We also to study the existence and uniqueness of coupled fixed points using monotone property instead of the often used mixed monotone property. Supporting example is also provided. An open problem is given at the end of the paper.***

Keywords: Coupled fixed point, mixed monotone property, monotone property, partial b - metric, complete partially ordered partial b - metric space, Geraghty - type contraction.

Mathematics Subject Classification (2010)}: 54H25, 47H10

1. Introduction

Banach [7] contraction principle is a powerful tool for solving many problems in applied mathematics and sciences. It has been improved and extended in many ways. Most of the fixed point theorems in nonlinear analysis usually start with Banach [7] contraction principle. A huge amount of literature is witnessed on applications, generalizations and extensions of this principle carried out by several authors in different directions like weakening the hypothesis and considering different mappings. Fixed point theory is an essential tool in the study of various varieties of problems in control theory, economic theory, nonlinear analysis and global analysis. In particular, Geraghty[11] proved in 1973 an interesting generalization of Banach [7] contraction principle. In 1989, Bakktin [6] introduced the concept of a b - metric space as a generalization of a metric space. In 1993, Czerwik[9] extended many results related to the b - metric space. In 1994, Matthews [17] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, O'Neill [26] generalized the concept of partial metric space by admitting negative distances. In 2013, Shukla [32] generalized both the concepts of b - metric and partial metric space by introducing the notation of partial b - metric spaces. Many authors [14, 33, 20, 30, 35] recently studied the existence of fixed points of self maps in different types of metric spaces. Some authors [4, 18, 22, 28, 29] obtained some fixed point theorems in b - metric spaces. Mustafa [22] gave a generalization of Banach contraction principle in complete ordered partial b - metric space. The notion of a coupled fixed point was introduced and studied by Opoitsev [25] and then by Guo and Lakshmikantham [12]. In 2006 Bhaskar and Lakshmikantham [5] introduced the notion of mixed monotone property and proved coupled fixed point theorems. They also studied and proved the following classical coupled fixed point theorems for mappings by using this property under contractive type conditions.

In this paper we prove the existence and uniqueness of coupled fixed points for contractive mappings controlled by a generalized contractive type condition on a complete partially ordered partial b- metric space under Geraghty - type contraction using monotone property instead of mixed monotone property. In fact, we show that coupled fixed point theorems on a complete partially ordered partial b - metric space with coefficient of partial b - metric space $s \geq 1$ by using monotone property under Geraghty - type contraction. A supporting example is also given. An open problem is given at the end of the paper.

2. Page Size and Layout

2.1 (S.Shukla[32]) Let X be a non empty set and let $s \geq 1$ be a given real number. A function $p: X \times X \to [0, \infty)$ is called a partial b - metric if for all $x, y, z \in X$ the following conditions are satisfied.

(i) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$

(ii) $p(x,x) \leq p(x,y)$

(iii) $p(x,y) = p(y,x)$

(iv) $p(x,y) \leq s\{p(x,z) + p(z,y)\} - p(z,z)$ The pair (X, p) is called a partial b - metric space. The number $s \geq 1$ is called a coefficient of (X, p).

2.2 (Mustafa et.al 22) A sequence $\{x_n\}$ in a partial b - metric space (X, p) is said to be:

I.(i) convergent if there exists a point $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = p(x,x)$

(ii) a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite

II. a partial b - metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that

$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x,x)$.

2.3 (E.Karapinar and B.Samet [15]) Let $(X, \leq)$ be a partially ordered set. A sequence $\{x_n\} \in X$ is said to be non decreasing, if $x_n \leq x_{n+1} \forall n \in N$

2.4 (Z.Mustafa[22]) A triple $(X, \leq, p)$ is called an ordered partial b - metric space if $(X, \leq)$ is a partially ordered set and p is a partial b - metric on X. For definiteness sake Sastry et.al [29](Definition 2.1) defined the triple $(X, \leq, p)$ as partially ordered partial b - metric space. A partially ordered partial metric space $(X, \leq, p)$ is said to be complete if every Cauchy sequence in X is convergent.

2.5 (T.G.Bhaskar and V.Lakshmikantham [5])Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. Then F is said to have the mixed monotone property if $F(x, y)$ is monotone non - decreasing in x and is monotone non - increasing in y i.e., for any

$x, y \in X$, $x_1, x_2 \in X$, $x_1 \leq x_2$

$\Rightarrow F(x_1, y) \leq F(x_2, y)$ (2.5.1)

and $y_1, y_2 \in X$, $y_1 \leq y_2$

$\Rightarrow F(x, y_1) \geq F(x, y_2)$ (2.5.2)

2.6 (Kadelburg et.al [13])Let $(X, \leq)$ be a partially ordered set. For $x, y \in X$, $x \geq y \Rightarrow y \leq x$.

Let $f, g : X \to X$ be mappings. f is said to be g - non decreasing (resp., g - non increasing) if,

$gx \leq gy \Rightarrow fx \leq fy$ (resp., $fy \leq fx$). If g is an identity mapping, then f is said to be non decreasing (resp., non increasing).

2.7 (Kadelburg et.al [13] Definition 2.1) Let $(X, \leq)$ be a partially ordered set and let $F : X \times X \to X$ and $g : X \to X$ be two mappings. The mapping F is said to have the g - monotone property if F is monotone g - non decreasing in both of its arguments, that is, for any $x, y \in X$,

$x_1, x_2 \in X$, $gx_1 \leq gx_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$ (2.7.1)

And

$y_1, y_2 \in X$, $gy_1 \leq gy_2 \Rightarrow F(x, y_1) \leq F(x, y_2)$ (2.7.2) hold.

2.8 (T.G.Bhaskar and V.Lakshmikantham [5]) Let X be a non empty set and $F : X \times X \to X$, $g : X \to X$ be two mappings. An element $(x, y) \in X \times X$ is said to be

(i) a coupled fixed point of the mapping F if $F(x, y) = x$ and $F(y, x) = y$

(ii) a coupled coincidence point of the mappings g and F if $F(x, y) = gx$ and $F(y, x) = gy$ and in this case (gx, gy) is called a coupled point of coincidence.

(iii) a common coupled fixed point of the mappings g and F if $F(x, y) = gx = x$ and $F(y, x) = gy = y$

2.9 (Choudhury, B.S et.al. [10]) Let (X, d) be a metric space and let $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then g, F are said to be compatible if

$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0$ and

$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0$ hold, whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n$ and

$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n$

2.10 (Kadelburg et.al [13]) Define $\Theta = \{\theta / \theta : [0,\infty) \times [0,\infty) \to [0,1)\}$ which satisfies the following conditions:

(i) $\theta(q,t) = \theta(t,q)$ $\forall q, t \in [0,\infty)$

(ii) for any two sequences $\{q_n\}$ and $\{t_n\}$ of non negative real numbers, $\theta(q_n, t_n) \to 1 \Rightarrow q_n, t_n \to 0$

2.11 (Kadelburg et.al [13] Theorem 3.1) Let $(X, \leq, d)$ be a complete partially ordered metric space and let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has the g - monotone property. Suppose the following hold:

(i) g is continuous and $g(X)$ is closed.

(ii) $F(X \times X) \subset g(X)$ and g and F are compatible.

(iii) There exists $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \leq F(y_0, x_0)$

(iv) there exists $\theta \in \Theta$ such that for all $x, y, u, v \in X$ satisfying $gx \leq gu$ and $gy \leq gv$ or $gx \geq gu$ and $gy \geq gv$,

$d(F(x,y), F(u,v)) \leq \theta(d(gx,gu), d(gy,gv))$ max $\{(d(gx,gu), d(gy,gv))\}$ (2.11.1) holds (v)(a) F is continuous (or) (b) if, for an increasing sequence $\{x_n\}$ in X, $x_n \to x \in X$ as $n \to \infty$, then $x_n \le x$ $\forall n \in N$

Then there exist $u, v \in X$ such that $gu = F(u,v)$ and $gv = F(v,u)$, that is g and F have a coupled coincidence point.

2.12 {Main Result} In this section we continue our study of extending the concepts proved by Kadelburg et.al[13] for partially ordered partial b - metric space $(X, \le, p)$. We study the existence and uniqueness of common coupled fixed point theorems for Geraghty - type contraction mappings using monotone property in partially ordered partial b - metric spaces. We begin this section with the following definition.

2.13{Definition} (Sastry et.al.[29]) Suppose $(X, \le)$ is a partially ordered set and p is a partial b - metric as in definition 1.1 with coefficient $s \ge 1$. Then we say that the triplet $(X, \le, p)$ is a partially ordered partial b - metric space. A partially ordered partial b - metric space $(X, \le, p)$ is said to be complete if every Cauchy sequence in X is convergent.

2.14 {Definition} We adopt the same definition given by Kadelburg.et.al[13] for Θ. Let $(X, \le, p)$ be a partially ordered partial b - metric space, with coefficient $s \ge 1$.Define $\Theta = \{\theta / \theta : [0,\infty) \times [0,\infty) \to [0,1)$ and s is the coefficient $(X, \le, p)\}$ which satisfies the following conditions:

(i) $\theta(q,t) = \theta(t,q)$ $\forall q,t \in [0,\infty)$

(ii) for any two sequences $\{q_n\}$ and $\{t_n\}$ of non negative real numbers,

$\theta(q_n, t_n) \to 1 \Rightarrow q_n, t_n \to 0$.

Now we state the following useful lemmas, whose proofs can be found in Sastry. et. al [29].

2.15{Lemma} Let $(X, \le, p)$ be a complete partially ordered partial b - metric space.

Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$. Then $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, y) = p(x,y)$ and hence $x = y$

2.16{Lemma} (i) $p(x,y) = 0 \Rightarrow x = y$

(ii) $\lim_{n\to\infty} (x_n, x) = 0 \Rightarrow p(x,x) = 0$ and hence $x_n \to x$ as $n \to \infty$

2.17{Theorem} Let $(X, \le, p)$ be a complete partially ordered partial b - metric space with coefficient $s \ge 1$ and let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has the g - monotone property. Suppose the following hold:

(i) g is continuous and $g(X)$ is closed.

(ii) $F(X \times X) \subset g(X)$ and g and F are compatible.

(iii) there exist $x_0, y_0 \in X$ such that $gx_0 \le F(x_0, y_0)$ and $gy_0 \le F(y_0, x_0)$

(iv) there exists $\theta \in \Theta$ such that for all $x, y, u, v \in X$ satisfying $gx \le gu$ and $gy \le gv$ or $gx \ge gu$ and $gy \ge gv$, That is (gx, gy) is comparable with (gu, gv) (see definition 2.21)

$sp(F(x,y), F(u,v)) \le \theta(p(gx,gu), p(gy,gv))$ max $\{(p(gx,gu), p(gy,gv))\}$ (2.17.0)

holds

(v)(a) F is continuous (or) (b) if, for an increasing sequence $\{x_n\}$ in X, $x_n \to x \in X$ as $n \to \infty$, then $x_n \le x$ $\forall n \in N$

Then there exist $u, v \in X$ such that $gu = F(u,v)$ and $gv = F(v,u)$, that is g and F have a coupled coincidence point.

Proof: Let $x_0, y_0 \in X$ be as in (iii). We have by (ii) $F(X \times X) \subset g(X)$

$\therefore \exists$ $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$

Then we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_n = F(x_{n-1}, y_{n-1})$ and $gy_n = F(y_{n-1}, x_{n-1})$ for $n = 1, 2, \ldots$

If $gx_n = gx_{n-1} = F(x_{n-1}, y_{n-1})$ and $gy_n = gy_{n-1} = F(y_{n-1}, x_{n-1})$ for some $n \in N$ then (gx_{n-1}, gy_{n-1}) is a coupled coincidence point of g and F

$\therefore$ we will assume that for each $n \in N$, $gx_n \ne gx_{n-1}$ or $gy_n \ne gy_{n-1}$ holds.

By (iii), $gx_0 \le F(x_0, y_0) = gx_1$ and $gy_0 \le F(y_0, x_0) = gy_1$

$\therefore$ the g - monotone property of F $\Rightarrow$

$gx_1 = F(x_0, y_0) \le F(x_1, y_0) \le F(x_1, y_1) = gx_2$ and

$gy_1 = F(y_0, x_0) \le F(y_1, x_0) \le F(y_1, x_1) = gy_2$

$\therefore$ by mathematical induction, $gx_{n-1} \le gx_n$ and $gy_{n-1} \le gy_n$ $\forall n \in N$ (2.17.1)

But by our assumption $gx_n \ne gx_{n-1}$ or $gy_n \ne gy_{n-1}$

$\therefore$ $gx_{n-1} < gx_n$ and $gy_{n-1} < gy_n$ $\forall n \in N$

Now by (2.17.0),

$$sp(gx_n, gx_{n+1}) = sp(F(x_{n-1}, y_{n-1}), F(x_n, y_n))$$
$$\leq \theta(p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)$$
$$\max(p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n) \quad (2.17.2)$$

and

$$sp(gy_n, gy_{n+1}) = sp(F(y_{n-1}, x_{n-1}), F(y_n, x_n))$$
$$\leq \theta(p(gy_{n-1}, gy_n), p(gx_{n-1}, gx_n)$$
$$\max(p(gy_{n-1}, gy_n), p(gx_{n-1}, gx_n)$$
$$= \theta(p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)$$
$$max\{(p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)\} \quad (2.17.3)$$

From (2.17.2) and (2.17.3), we get

$$max\{(p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\}$$
$$\leq s\ max\{(p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\}$$
$$\leq \theta(p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)$$
$$max\{(p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)\}$$
$$\leq max\{(p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)\}\ \forall n \in N$$

(2.17.4)

Let $J_n = max\{(p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)\}$.

Then by (2.17.4), the sequence $\{J_n\}$ is decreasing.

It follows that sequence $\{J_n\}$ is convergent.

$\therefore \exists$ some $J \geq 0$ such that $\lim_{n\to\infty} J_n = J$.

We claim that $J = 0$. Let us assume that J > 0 we have from (2.17.4)

$$\frac{J_{n+1}}{J_n} = \frac{max\{(p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\}}{max\{(p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)\}}$$
$$\leq \theta(p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n))$$

< 1 (clearly by lemma 2.16 (i) $J_n \neq 0$)

Allowing $n \to \infty$,

we get, $\lim_{n\to\infty} \theta(p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)) = 1$

$\therefore$ By definition 2.14 (ii), we have

$\lim_{n\to\infty} p(gx_{n-1}, gx_n) = 0$ and

$\lim_{n\to\infty} p(gy_{n-1}, gy_n)) = 0$ (2.17.5)

$\therefore \lim_{n\to\infty} J_n = \lim_{n\to\infty} (p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)) = 0$,

a contradiction.

$\therefore J = \lim_{n\to\infty} (p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)) = 0$ (2.17.6)

Let us show that the sequences $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

If possible let $\{gx_n\}$ or $\{gy_n\}$ fails to be a Cauchy sequence, Then either $\lim_{m,n\to\infty} p(gx_m, gx_n) \neq 0$ or

$\lim_{m,n\to\infty} p(gy_m, gy_n) \neq 0$

$\therefore max\{\lim_{m,n\to\infty} p(gx_m, gx_n), \lim_{m,n\to\infty} p(gy_m, gy_n)\} \neq 0$

$\therefore \exists ò > 0$ for which we can find sub sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k > m_k > k$ å

$max\ \{p\ (gx_{m_k}, gx_{n_k}), p\ (gy_{m_k}, gy_{n_k})\} \geq ò$.

Further we can choose n_k to be the smallest positive integer å $\{m_k\}$ and $\{n_k\}$

satisfy $max\ \{p\ (gx_{m_k}, gx_{n_k}), p\ (gy_{m_k}, gy_{n_k})\} \geq ò$ and

$max\ \{p\ (gx_{m_k}, gx_{n_k-1}), p\ (gy_{m_k}, gy_{n_k-1})\} < ò$

Now we explore the properties of these squences which we use insequel.

I. $\lim_{k\to\infty} (max\ \{p\ (gx_{m_k}, gx_{n_k}), p\ (gy_{m_k}, gy_{n_k})\}) = ò$

II

ò

$$\leq s^2 \liminf_{k\to\infty} (max\ \{p\ (gx_{m_k-1}, gx_{n_k-1}), p\ (gy_{m_k-1}, gy_{n_k-1})\})$$
$$\leq s^2 \limsup_{k\to\infty} (max\ \{p\ (gx_{m_k-1}, gx_{n_k-1}), p\ (gy_{m_k-1}, gy_{n_k-1})\})$$
$$\leq s^3 ò$$

III.

$$ò \leq s \liminf_{k\to\infty} (max\ \{p\ (gx_{m_k}, gx_{n_k-1}), p\ (gy_{m_k}, gy_{n_k-1})\})$$
$$\leq s \limsup_{k\to\infty} (max\ \{p\ (gx_{m_k}, gx_{n_k-1}), p\ (gy_{m_k}, gy_{n_k-1})\}) \leq sò$$

Now $sp\ (gx_{m_k+1}, gx_{n_k}) =$

$$sp\ ((F(x_{m_k}, y_{m_k}), F(x_{n_k-1}, y_{n_k-1}))$$
$$\leq \theta(p(gx_{m_k}, gx_{n_k-1}), p(gy_{m_k}, gy_{n_k-1}))$$
$$(max\ \{p(gx_{m_k}, gx_{n_k-1}), p(gy_{m_k}, gy_{n_k-1})\})$$
$$\leq\ max\ \{p(gx_{m_k}, gx_{n_k-1}), p(gy_{m_k}, gy_{n_k-1})\} \quad (2.17.7)$$

Similarly $sp\ (gy_{m_k+1}, gy_{n_k}) \leq$

$$max\{p(gy_{m_k}, gy_{n_k-1}), p(gx_{m_k}, gx_{n_k-1}) \quad (2.17.8)$$

$\therefore$ from (2.17.7) and (2.17.8) we have

$$s\,max\{p\ (gx_{m_k+1}, gx_{n_k}), p\ (gy_{m_k+1}, gy_{n_k})\} \leq$$
$$max\{p(gx_{m_k}, gx_{n_k-1}), p(gy_{m_k}, gy_{n_k-1})\} < ò \quad (2.17.9)$$

Now, $p\ (gx_{m_k}, gx_{n_k}) \leq s(p\ (gx_{m_k}, gx_{m_k+1})$

$+\ p\ (gx_{m_k+1}, gx_{n_k}))\ -\ p\ (gx_{m_k+1}, gx_{m_k+1})$

$\leq\ s(p\ (gx_{m_k}, gx_{m_k+1})\ +\ p\ (gx_{m_k+1}, gx_{n_k}))$

$\leq\ sp\ (gx_{m_k}, gx_{m_k+1})\ +\ ò$ (by (2.17.9))

Similarly,

$p\ (gy_{m_k}, gy_{n_k})\ \leq\ sp\ (gy_{m_k+1}, gy_{m_k})\ +\ ò$ (by (2.17.9))

$\therefore\ ò \leq\ max\ \{p\ (gx_{m_k}, gx_{n_k}), p\ (gy_{m_k}, gy_{n_k})\}$

$\leq\ s\,max\ \{p\ (gx_{m_k+1}, gx_{m_k}), p\ (gy_{m_k+1}, gy_{m_k})\} + ò$

Allowing $k \to \infty$ and using (2.17.5)

$ò \le max\,\{p\,(gx_{m_k}, gx_{n_k}), p\,(gy_{m_k}, gy_{n_k})\} \le ò$

$\therefore \lim_{k\to\infty} (max\,\{p\,(gx_{m_k}, gx_{n_k}), p\,(gy_{m_k}, gy_{n_k})\})$ exists and equal to $ò$

Therefore (I) follows.

Also we have $p\,(gx_{m_k}, gx_{n_k}) \le s(p\,(gx_{m_k}, gx_{n_k-1}) +$

$p\,(gx_{n_k-1}, gx_{n_k})) - p\,(gx_{n_k-1}, gx_{n_k-1})$

$\le s(s(p\,(gx_{m_k}, gx_{m_k-1}) + p\,(gx_{m_k-1}, gx_{n_k-1}))$ -

$p\,(gx_{m_k-1}, gx_{m_k-1})) + sp\,(gx_{n_k-1}, gx_{n_k})$

$\le s^2 p\,(gx_{m_k}, gx_{m_k-1}) + s^2 p\,(gx_{m_k-1}, gx_{n_k-1}) +$

$sp\,(gx_{n_k-1}, gx_{n_k})$

Similarly $p\,(gy_{m_k}, gy_{n_k}) \le s^2 p\,(gy_{m_k}, gy_{m_k-1}) +$

$s^2 p\,(gy_{m_k-1}, gy_{n_k-1}) + sp\,(gy_{n_k-1}, gy_{n_k})$

$\therefore ò \le max\,\{p\,(gx_{m_k}, gx_{n_k}), p\,(gy_{m_k}, gy_{n_k})\}$

$\le s^2 max\,\{p\,(gx_{m_k}, gx_{m_k-1}), p\,(gy_{m_k}, gy_{m_k-1})\} +$

$s^2 max\,\{p\,(gx_{m_k-1}, gx_{n_k-1}), p\,(gy_{m_k-1}, gy_{n_k-1})\} +$

$s\,max\,\{p\,(gx_{n_k}, gx_{n_k-1}), p\,(gy_{n_k}, gy_{n_k-1})\}$

Allowing $k \to \infty$ and using (2.17.5)

we have $ò \le$

$s^2 \liminf_{k\to\infty}$

$(max\,\{p\,(gx_{m_k-1}, gx_{n_k-1}), p\,(gy_{m_k-1}, gy_{n_k-1})\})$ (2.17.10)

On the other hand,

$p\,(gx_{m_k-1}, gx_{n_k-1}) \le s(p\,(gx_{m_k-1}, gx_{m_k})$

$+ p\,(gx_{m_k}, gx_{n_k-1}))$ - $p\,(gx_{m_k}, gx_{n_k})$

$\le s(p\,(gx_{m_k-1}, gx_{m_k}) + p\,(gx_{m_k}, gx_{n_k-1}))$

$\le sp\,(gx_{m_k-1}, gx_{m_k}) + sò$

Similarly $p\,(gy_{m_k-1}, gy_{n_k-1}) \le sp\,(gy_{m_k-1}, gy_{m_k}) + sò$

$\therefore max\,\{p\,(gx_{m_k-1}, gx_{n_k-1}), p\,(gy_{m_k-1}, gy_{n_k-1})\}$

$\le s\,max\,\{p\,(gx_{m_k-1}, gx_{m_k}), p\,(gy_{m_k-1}, gy_{m_k})\} + sò$

(2.17.11)

Allowing $k \to \infty$ and using (2.17.5)

$\limsup_{k\to\infty}$

$(max\,\{p\,(gx_{m_k-1}, gx_{n_k-1}), p\,(gy_{m_k-1}, gy_{n_k-1})\}) \le sò$

(2.17.12)

$\therefore$ From (2.17.10) and (2.17.12)

$ò \le s^2 \liminf_{k\to\infty}$

$(max\,\{p\,(gx_{m_k-1}, gx_{n_k-1}), p\,(gy_{m_k-1}, gy_{n_k-1})\})$

$\le s^2 \limsup_{k\to\infty}$

$(max\,\{p\,(gx_{m_k-1}, gx_{n_k-1}), p\,(gy_{m_k-1}, gy_{n_k-1})\}) \le s^3 ò$

Therefore (II) follows.

Finally we have, $p\,(gx_{m_k}, gx_{n_k}) \le s(p\,(gx_{m_k}, gx_{n_k-1}) +$

$p\,(gx_{n_k-1}, gx_{n_k}))$

$\le sò + sp\,(gx_{n_k-1}, gx_{n_k}))$

Similarly $p\,(gy_{m_k}, gy_{n_k}) \le s(p\,(gy_{m_k}, gy_{n_k-1}) +$

$p\,(gy_{n_k-1}, gy_{n_k}))$

$\le sò + sp\,(gy_{n_k-1}, gy_{n_k}))$

$\therefore ò \le max\,\{p\,(gx_{m_k}, gx_{n_k}), p\,(gy_{m_k}, gy_{n_k})\} \le$

$s\,max\,\{p\,(gx_{m_k}, gx_{n_k-1}), p\,(gy_{m_k}, gy_{n_k-1})\} +$

$s\,max\,\{p\,(gx_{n_k-1}, gx_{n_k}), p\,(gy_{n_k-1}, gy_{n_k})\}$

$\le sò + s\,max\,\{p\,(gx_{n_k-1}, gx_{n_k}), p\,(gy_{n_k-1}, gy_{n_k})\}$

Allowing $k \to \infty$ and using (2.17.5)

$ò \le s \liminf_{k\to\infty} (max\,\{p\,(gx_{m_k}, gx_{n_k-1}), p\,(gy_{m_k}, gy_{n_k-1})\})$

$\le s \limsup_{k\to\infty} (max\,\{p\,(gx_{m_k}, gx_{n_k-1}), p\,(gy_{m_k}, gy_{n_k-1})\}) \le sò$

Therefore (III) follows.

Now from (2.17.0)

$sp\,(gx_{m_k}, gx_{n_k}) = sp\,(F(x_{m_k-1}, y_{m_k-1}), F(x_{n_k-1}, y_{n_k-1}))$

$\le \theta(p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1}))$

$(max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\})$

$\le max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\}$ (2.17.13)

Similarly

$sp\,(gy_{m_k}, gy_{n_k}) = sp\,(F(y_{m_k-1}, x_{m_k-1}), F(y_{n_k-1}, x_{n_k-1}))$

$\le \theta(p(gy_{m_k-1}, gy_{n_k-1}), p(gx_{m_k-1}, gx_{n_k-1}))$

$max(\{p\,(g(y_{m_k-1}, gy_{n_k-1}), g(x_{m_k-1}, gx_{n_k-1})\})$

$\le max\{p(gy_{m_k-1}, gy_{n_k-1}), p(gx_{m_k-1}, gx_{n_k-1})$ (2.17.14)

$\therefore$ from (2.17.13) and (2.17.14) we have

$sò \le s\,max\{p\,(gx_{m_k}, gx_{n_k}), p\,(gy_{m_k}, gy_{n_k})\}$

$\le max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})$

$= t_k$ (2.17.15)

where $t_k = max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})$

$\therefore sò \le t_k$ for every k

But by(II), $\liminf_{k\to\infty} (t_k) \le \limsup_{k\to\infty} (t_k) \le sò$

Hence $sò \le \liminf_{k\to\infty} (t_k) \le \limsup_{k\to\infty} (t_k) \le sò$

$\therefore \lim_{k\to\infty} (t_k) = sò$

From (2.17.0), (2.17.15),

$sò \le s\,max\{p\,(gx_{m_k}, gx_{n_k}), p\,(gy_{m_k}, gy_{n_k})\}$

$\le \theta(p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1}))$

$(max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\}) \le t_k$

Allowing $k \to \infty$, then

$sò \leq sò \lim_{k\to\infty} \theta(p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1}))$

$\leq sò$

$\Rightarrow \lim_{k\to\infty} \theta(p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})) = 1$

$\Rightarrow \lim_{k\to\infty} p(gx_{m_k-1}, gx_{n_k-1}) = 0$ and

$\lim_{k\to\infty} p(gy_{m_k-1}, gy_{n_k-1}) = 0$

$\therefore \lim_{k\to\infty} t_k =$

$\lim_{k\to\infty}(max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\}) = 0$, which is a contradiction.

$\therefore$ sequences $\{gx_n\},\{gy_n\}$ are Cauchy sequences.

$\therefore$ by (2.17.5), $\lim_{m,n\to\infty} p(gx_m, gx_n) = 0$ and $\lim_{m,n\to\infty} p(gy_m, gy_n) = 0$

Since $g(X)$ is closed $\exists x, y$ such that $\{gx_n\} \to gx$ and $\{gy_n\} \to gy$ as $n \to \infty$

$\therefore \lim_{n\to\infty} p(gx_n, gx) = 0 = p(gx, gx)$ and $\lim_{n\to\infty} p(gy_n, gy) = 0 = p(gy, gy)$ (2.17.16)

We have $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ and

Let 2.17 (v) (a) holds $\Rightarrow$ F is continuous

$\therefore \lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(x, y)$ and also $\lim_{n\to\infty} gx_{n+1} = gx$

$\therefore$ By lemma 2.15 and (2.17.5), we have $gx = F(x, y)$

Similarly $gy = \lim_{n\to\infty} gy_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(y, x)$

$\therefore$ gx and gy are coupled points of coincidence of F in X.

Suppose 2.5 $(v)(b)$ holds.

Since $\{gx_n\}$ and $\{gy_n\}$ are increasing sequences (by (2.17.1))

and since we have $\{gx_n\} \to gx$ and $\{gy_n\} \to gy$ as $n \to \infty$, from (b)

$gx_n \leq gx$ and $gy_n \leq gy$ for each $n \in N$

Now by triangle inequality and (2.17.0),

$p(F(x, y), gx) \leq s\{p(F(x, y), gx_{n+1}) +$

$p(gx_{n+1}, gx)\} - p(gx_{n+1}, gx_{n+1})$

$\leq s\{p(F(x, y), gx_{n+1}) + p(gx_{n+1}, gx)\}$

$= s\{p(F(x, y), F(x_n, y_n)) + p(gx_{n+1}, gx)\}$

$\leq \theta(p(gx, gx_n), p(gy, gy_n))$

$(max\{p(gx, gx_n), p(gy, gy_n)\}) + sp(gx_{n+1}, gx)$

Allowing $n \to \infty$, by (2.17.16) $p\Big(F(x, y), gx\Big) = 0$

Therefore by lemma2.15

$gx = F(x, y)$ and similarly $gy = F(y, x)$.

Hence gx and gy are coupled points of coincidence of F in X.

2.18 Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$ and let $F: X \times X \to X$ be a mapping such that F has the monotone property. Suppose the following hold:

(i) there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$

(ii) there exists $\theta \in \Theta$ such that for all $x, y, u, v \in X$ satisfying $x \leq u$ and $y \leq v$ or $u \leq x$ and $v \leq y$,

$sp(F(x, y), F(u, v)) \leq \theta(p(x, u), p(y, v))$ max $\{(p(x, u), p(y, v))\}$ (2.18.1)

holds

(iii)(a) F is continuous (or)

(b) if, for an increasing sequence $\{x_n\}$ in X, $x_n \to x \in X$ as $n \to \infty$, then $x_n \leq x$ $\forall n \in N$

Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is F has a coupled fixed point.

Proof: Putting $g = I$ where I is an identity map on X, then result follows from Theorem 2.17.

2.19 Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$ and let $F: X \times X \to X$ and $g: X \to X$ be two mappings such that F has the g - monotone property. Suppose the following holds:

(i) g is continuous and $g(X)$ is closed.

(ii) $F(X \times X) \subset g(X)$ and g and F are compatible.

(iii) there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \leq F(y_0, x_0)$

(iv) there exists $k \in [0,1)$ such that for all $x, y, u, v \in X$ satisfying $gx \leq gu$ and $gy \leq gv$ or $gu \leq gx$ and $gv \leq gy$,

$sp(F(x, y), F(u, v)) \leq k \max\{(p(gx, gu), p(gy, gv))\}$ (2.19.1)

holds

(v)(a) F is continuous (or)

(b) if, for an increasing sequence $\{x_n\}$ in X, $x_n \to x \in X$ as $n \to \infty$, then $x_n \leq x$ $\forall n \in N$

Then there exists $u, v \in X$ such that $gu = F(u, v)$ and $gv = F(v, u)$, that is g and F have a coupled coincidence point.

Proof: Taking $\theta(t_1, t_2) = k$ $\forall t_1, t_2 \in [0, \infty)$, then result follows from Theorem 2.17.

2.20 Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$ and let $F : X \times X \to X$ be a mapping such that

(i) F has the monotone property.

(ii) there exist $x_0, y_0 \in X$ such that

$x_0 \leq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$

(iii) there exists $k \in [0,1)$ such that for all $x, y, u, v \in X$ satisfying $x \leq u$ and $y \leq v$ or $u \leq x$ and $v \leq y$,

$sp(F(x, y), F(u,v)) \leq k$ max $\{(p(x,u), p(y,v))\}$

holds

(v)(a) F is continuous (or)

(b) if, for an increasing sequence $\{x_n\}$ in X,

$x_n \to x \in X$ as $n \to \infty$, then $x_n \leq x \;\; \forall n \in N$

Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is g and F have a coupled coincidence point.

Proof: Taking $\theta(t_1, t_2) = k \quad \forall t_1, t_2 \in [0, \infty)$, and Putting $g = I$ where I is an identity map on X, then result follows from Theorem 2.17.

Now, we obtain a sufficient condition for the uniqueness of coupled fixed point. For this we need the following definition.

2.21{Definition} Suppose (x, y) and (u,v) $\in X \times X$.

We write $(x, y) \leq (u,v)$ if $x \leq u$ and $y \leq v$.

Similarly $(x, y) \geq (u,v)$ if $x \geq u$ and $y \leq v$.

If $(x, y) \leq (u,v)$ or $(x, y) \geq (u,v)$ then we say that (x, y) and (u,v) are comparable in $X \times X$. We observe that $\leq$ is a partial order in $X \times X$.

2.22 {Now we state and prove our second main result.}

In this result we obtain a sufficient condition for the existence and uniqueness of common coupled fixed point.

{Theorem}: Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$ and let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has the g - monotone property. Suppose the following holds:

(i) g is continuous and $g(X)$ is closed.

(ii) $F(X \times X) \subset g(X)$ and g and F are compatible.

(iii) there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \leq F(y_0, x_0)$

(iv) there exists $\theta \in \Theta$ such that for all $x, y, u, v \in X$ satisfying $gx \leq gu$ and $gy \leq gv$ or $gx \geq gu$ and $gy \geq gv$,

$sp(F(x, y), F(u,v)) \leq \theta(p(gx, gu), p(gy, gv))$ max $\{(p(gx, gu), p(gy, gv))\}$ (2.22.1)

holds

(v)(a) F is continuous (or)

(b) if, for an increasing sequence $\{x_n\}$ in X, $x_n \to x \in X$ as $n \to \infty$, then $x_n \leq x \;\; \forall n \in N$

Then there exist $u, v \in X$ such that $gu = F(u,v)$ and $gv = F(v,u)$, that is g and F have a coupled coincidence point.

(vi)Further suppose for every (x, y) and (u,v) in $X \times X$, there exist (w, z) such that (gw, gz) is comparable with (gx, gy), (gu, gv)

Then g and F have a unique common coupled fixed point.

Proof: We have by theorem 2.5 the set of coupled coincidence points is non-empty. Suppose (x, y) and (u,v) are coupled coincidence points of the mappings $F : X \times X \to X$ and $g : X \to X$, Then, $gx = F(x, y)$, $gy = F(y, x)$ and $gu = F(u,v)$, $gv = F(v,u)$. By assumption, there exists (w, z) in $X \times X$ such that (gx, gy) and (gu, gv) are comparable with (gw, gz).

We may assume without loss of generality that $(gx, gy) \leq (gw, gz)$

so that $gx \leq gw$ and $gy \leq gz$ (2.22.2)

Let w_{0} = w and $z_0 = z$.

Write $gw_1 = F(w_0, z_0)$ and $gz_1 = F(z_0, w_0)$. Thus inductively we can define sequences $\{gw_n\}$ and $\{gz_n\}$ as $gw_n = F(w_{n-1}, z_{n-1})$ and $gz_n = F(z_{n-1}, w_{n-1})$ $\forall n \in N$

Now $gx = F(x, y) \leq F(w_0, z_0) = gw_1$

(since F is $g-$ monotonic, from (2.22.2))

$\therefore \; gx \leq gw_1$

Similarly $gy = F(y, x) \leq F(z_0, w_0) = gz_1$

(Since F is $g-$ monotonic, from (2.22.2))

$\therefore \; gy \leq gz_1$

Hence inductively $gx \leq gw_n$ and $gy \leq gz_n \;\; \forall n \in N$

$\therefore \; (gw_{n+1}, gz_{n+1}) \geq (gx, gy)$

$\therefore$ by applying (2.22.1) for gx and gw_{n+1}, we have,

$sp(gx, gw_{n+1}) = sp(F(x, y), F(w_n, z_n))$
$\leq \theta(p(gx, gw_n), p(gy, gz_n))$ max
$\{p(gx, gw_n), p(gy, gz_n)\}$

and

$sp(gy, gz_{n+1}) = sp(F(y, x), F(z_n, w_n))$
$\leq \theta(p(gy, gz_n), p(gx, gw_n))$ max
$\{p(gy, gz_n), p(gx, gw_n)\}$
$= \theta(p(gx, gw_n), p(gy, gz_n))$ max
$\{p(gx, gw_n), p(gy, gz_n)\}$
$\therefore \; max\{p(gx, gw_{n+1}), p(gy, gz_{n+1})\}$

$\leq \; s\,max\{p(gx, gw_{n+1}), p(gy, gz_{n+1})\}$

$\leq \theta(p(gx, gw_n), p(gy, gz_n))$ max

$\{p(gx, gw_n), p(gy, gz_n)\}$

$\leq$ max $\{p(gx, gw_n), p(gy, gz_n)\}$ (2.22.3)

Let $J_n = max\{p(gx, gw_n), p(gy, gz_n)\}$ so that sequence $\{J_n\}$ is decreasing and converges to J (say)

$\therefore \lim_{n\to\infty} J_n = J$

Let us suppose J > 0. Then by (2.22.3)

$$\frac{max\{p(gx, gw_{n+1}), p(gy, gz_{n+1})\}}{max\{p(gx, gw_n), p(gy, gz_n)\}}$$

$\leq \theta(p(gy, gz_n), p(gx, gw_n)) < 1$

Allowing $n \to \infty$, we have,

$\lim_{n\to\infty} \theta(p(gy, gz_n), p(gx, gw_n)) = 1$ (2.22.4)

$\Rightarrow \lim_{n\to\infty} p(gx, gw_n) = 0$ and $\lim_{n\to\infty} p(gy, gz_n) = 0$

$\therefore \lim_{n\to\infty}$ max $\{p(gx, gw_n), p(gy, gz_n)\} = 0$

which is a contradiction.

$\therefore J = \lim_{n\to\infty}$ max $\{p(gx, gw_n), p(gy, gz_n)\} = 0$

and hence $\lim_{n\to\infty} p(gx, gw_n) = 0$ and $\lim_{n\to\infty} p(gy, gz_n) = 0$ (2.22.5)

so that $gw_n \to gx$ and $gz_n \to gy$ as $n \to \infty$

(by lemma 2.16 (ii))

Similarly $\lim_{n\to\infty} p(gu, gw_n) = 0$ and

$\lim_{n\to\infty} p(gv, gz_n) = 0$ (2.10.6)

so that $gw_n \to gu$ and $gz_n \to gv$ as $n \to \infty$

(by lemma 2.16 (ii))

Now $p(gx, gu)$

$\leq \; s\{p(gx, gw_n) + p(gw_n, gu)\}$ - $p(gw_n, gw_n)$

$\leq \; s\{p(gx, gw_n) + p(gw_n, gu)\}$

Similarly $p(gy, gv) \; \leq \; s\{p(gy, gz_n) + p(gz_n, guv\}$

$\therefore$ from (2.22.3), (2.22.4)

$p(gx, gu) = 0$, $p(gy, gv) = 0$

$\therefore$ by lemma 2.16 (i)

$gx = gu$ and $gy = gv$

Hence we conclude that if (x, y) and (u, v) are coupled coincidence points of the mappings $F: X \times X \to X$ and $g: X \to X$ having a comparable point, then

$gx = gu$ and $gy = gv$ (2.22.7)

Further let

$l = gx = F(x, y)$ and $m = gy = F(y, x)$ (2.22.8)

Then $gl = g(gx) = gF(x, y)$ and

$gm = g(gy) = gF(y, x)$ (2.22.9)

But by the definitions of sequences $\{gw_n\}$ and $\{gz_n\}$ we have,

$\therefore \lim_{n\to\infty} F(w_n, z_n) = \lim_{n\to\infty} gw_{n+1} = \lim_{n\to\infty} gw_n$

$= gx = F(x, y)$ (2.22.10)

and

$\lim_{n\to\infty} F(z_n, w_n) = \lim_{n\to\infty} gz_{n+1} = \lim_{n\to\infty} gz_n$

$= gy = F(y, x)$ (2.22.11)

Since g and F are compatible, we have

$\lim_{n\to\infty} p(gF(w_n, z_n), F(gw_n, gz_n)) = 0$ and

$\lim_{n\to\infty} p(gF(z_n, w_n), F(gz_n, gw_n)) = 0$

$\therefore \; p(gF(x, y), F(gx, gy)) = 0$

$\Rightarrow gF(x, y) = F(gx, gy)$ (by lemma 2.16, from (2.22.10) and by $(i), (v)(a)$ of hypothesis)

$\therefore \; gl = gF(x, y) = F(gx, gy) = F(l, m)$ and

$\therefore \; p(gF(y, x), F(gy, gx)) = 0$

$\Rightarrow gF(y, x) = F(gy, gx)$

(by lemma 2.16, from (2.22.11) and by $(i), (v)(a)$ of hypothesis)

$\therefore \; gm = gF(y, x) = F(gy, gx) = F(m, l)$

Suppose condition $v(b)$ of hypothesis holds

Since $gw_n \to gx$ and $gz_n \to gy$ as $n \to \infty$

$\Rightarrow \; gw_n \leq gx$ and $gz_n \leq gy$

$\therefore$ by (2.17.0),

$sp(F(gx, gy), F(gw_n, gz_n)$

$\leq \theta(p(gx, gw_n), p(gy, gz_n))$

max$\{(p(gx, gw_n), p(gy, gz_n))\}$

$\leq$ max $\{(p(gx, gw_n), p(gy, gz_n))\} \to 0$ as $n \to \infty$

$\therefore \; p(F(gx, gy), F(gw_n, gz_n)) = 0$ as $n \to \infty$

$\Rightarrow \lim_{n\to\infty} F(gw_n, gz_n) = F(gx, gy)$

Similarly $p(F(gy, gx), F(gz_n, gw_n)) = 0$ as $n \to \infty$

$\Rightarrow \lim_{n\to\infty} F(gz_n, gw_n) = F(gy, gx)$

$\therefore \; p(gF(x, y), F(gx, gy)) = 0$

Again by the above discussion,

$gl = gF(x, y) = F(gx, gy) = F(l, m)$ and

$gm = gF(y, x) = F(gy, gx) = F(m, l)$

$\Rightarrow \; (l, m)$ is a coupled coincidence point.

Now, by hypothesis (vi), (gx, gy) and (gl, gm) have a comparable point

$\therefore$ by (2.22.7) and (2.22.8) $gl = gx = l$ and

$gm = gy = m$ (2.22.12)

$\therefore \; l = gl = F(l, m)$ and $m = gm = F(m, l)$

Hence, (l, m) is a common coupled fixed point of g and F.

Suppose (r, t) is a common coupled fixed point of g and F.

Again, by hypothesis (vi), (gl, gm) and (gr, gt) have a comparable point in $X \times X$.
Hence, by (2.22.7),
$l = gl = gr = r$ and $m = gl = gt = t$ (2.22.13)
$\therefore$ by (2.22.12) and (2.22.13) we conclude that $l = t$ and $m = t$.
Thus $(r, t) = (l, m)$
Hence g and F have a unique common coupled fixed point.
Condition (vi) of theorem 2.22 holds if $g(X) \times g(X)$ is a lattice under the partial order defined in 2.21

2.23 corollary: In addition to the hypothesis of corollary 2.20 assume that
(vi)for every (x, y) and (u, v) in $X \times X$, there exists (w, z) such that (gw, gz) is comparable with (gx, gy) and (gu, gv).
Then g and F has a unique common coupled fixed point.
Now we give an example in support of theorem 2.17

2.24 example: Let $X = [0,1]$ with usual ordering.
Define $p(x, y) = (max\{x, y\})^2 \ \forall x, y \in X$.
Clearly, $(X, \leq, p)$ is a partially ordered partial b - metric space with coefficient $s = 2$ (Mukheimer. A [21])
Define $F : X \times X \to X$ by
$$F(x, y) = \frac{1}{2} xy \text{ if } \forall x, y \in X$$
Define $\theta : [0, \infty) \times [0, \infty) \to [0,1)$ by
$$\theta(u, v) = \begin{cases} \dfrac{ln\{1 + (u + v)\}}{u + v} & \text{if } u > 0 \, or \, v > 0 \\ 0 \text{ if } u = 0 = v \end{cases}$$
Define $g(t) = t^2$
For $x, y, u, v \in X$,
$gx \leq gu$ and $gy \leq gv \Rightarrow x^2 \leq u^2$ and $y^2 \leq v^2$
$\therefore \ 2p(F(x, y), F(u, v)) = 2(\max\{F(x, y), F(u, v)\})^2$
$= 2max(\{\frac{1}{2}xy\}^2, \{\frac{1}{2}uv\}^2)$
$= \frac{1}{2}\{uv\}^2$ (2.24.1)
Now
$\theta(p(gx, gu), p(gy, gv)) \, max\{p(gx, gu), p(gy, gv)\}$
$= \theta(max\{x^2, u^2\}, max\{y^2, v^2\})$
$(max\{x^2, u^2\}, max\{y^2, v^2\})$
$= \theta(u^2, v^2) \max (u^2, v^2)$
$= \{\frac{ln\{1 + (u^2 + v^2)\}}{u^2 + v^2}\} \max (u^2, v^2)$
$= \{\frac{ln\{1 + (u^2 + v^2)\}}{u^2 + v^2}\} (v^2)$ when $u^2 \leq v^2$
Let us consider $f(x) = x - 2ln(1 + x)$
$$f'(x) = 1 - 2(\frac{1}{1 + x}) = \frac{x - 1}{x + 1} \leq 0 \text{ for } x \in X = [0,1]$$
$\therefore$ f is decreasing and $f(0) = 0$
$\therefore$ for $x \geq 0 \Rightarrow f(x) \leq f(0) = 0$
$\therefore$ $x \leq 2ln(1 + x)$ (2.24.2)
Since $u \in X = [0,1]$
$\Rightarrow u^2(u^2 + v^2) \leq (u^2 + v^2)$ and
$(u^2 + v^2) \leq 2ln\{1 + (u^2 + v^2)\}$ (from (2.24.2))
$\therefore$ $u^2(u^2 + v^2) \leq 2ln\{1 + (u^2 + v^2)\}$
$\Rightarrow \frac{1}{2}\{uv\}^2 \leq \{\frac{ln\{1 + (u^2 + v^2)\}}{u^2 + v^2}\} (v^2)$
when $u^2 \leq v^2$ (2.24.3)
Similarly $\frac{1}{2}\{uv\}^2 \leq \{\frac{ln\{1 + (u^2 + v^2)\}}{u^2 + v^2}\} (u^2)$
when $v^2 \leq u^2$
$\therefore$ $2p(F(x, y), F(u, v)) \leq \theta(p(gx, gu), p(gy, gv)) \, max\{p(gx, gu), p(gy, gv)\}$
$\therefore$ $sp(F(x, y), F(u, v)) \leq \theta(p(gx, gu), p(gy, gv)) \, max\{p(gx, gu), p(gy, gv)\}$
$\therefore$ the condition (2.17.0) holds.
(i) Clearly $g(x) = x^2 \ \forall x \in X$ is continuous and $g(X) = [0,1]$ is closed.
(ii) Let sequences $\{x_n\}$, $\{y_n\} \in X$ be such that
$\lim_{n \to \infty} gx_n = \lim_{n \to \infty} F(x_n, y_n) = a$ (say) and
$\lim_{n \to \infty} gy_n = \lim_{n \to \infty} F(y_n, x_n) = b$ (say)
$\Rightarrow \lim_{n \to \infty} (x_n)^2 = \lim_{n \to \infty} \frac{1}{2} x_n y_n$ and
$\lim_{n \to \infty} (y_n)^2 = \lim_{n \to \infty} \frac{1}{2} x_n y_n$
$\Rightarrow a = b = 0$
$\Rightarrow \lim_{n \to \infty} x_n = 0$ and $\lim_{n \to \infty} y_n = 0$
Now $g(F(x_n, y_n)) = g(\frac{1}{2} x_n y_n) = (\frac{1}{2} x_n y_n)^2$ and
$F(gx_n, gy_n) = \frac{1}{2}(x_n y_n)^2$
$\therefore p(g(F(x_n, y_n)), F(gx_n, gy_n)) =$
$\max\{(\frac{1}{2} x_n y_n)^2, \frac{1}{2}(x_n y_n)^2\} = \frac{1}{2}(x_n y_n)^2$
$\therefore \lim_{n \to \infty} p(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$
Similarly, $\lim_{n \to \infty} p(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0$

$\therefore$ g and F are compatible.

Further $F(X \times X) = [0, \frac{1}{2}] \subset [0,1] = g(X)$

(iii) Taking $x_0 = 0, y_0 = 0 \Rightarrow$

$F(x_0, y_0) = 0 = gx_0, F(y_0, x_0) = 0 = gy_0$

(iv) F is continuous.

$\therefore$ F satisfies the hypothesis of theorem 2.17

Now $F(0,0) = g0 = 0$ and $F(0,0) = g0 = 0$

Hence (0, 0) is a coupled coincidence point of g and F in X

{Open Problem}: Are the theorems 2.17, 2.22 and their corollaries true if the contractive condition defined independent of the coefficient s of complete partially ordered partial b - metric space $(X, \leq, p)$

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