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On Generalized S Topological Groups

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Abstract: ***The generalized topological group is introduced by Muard Hussain et al. [5]. The authors Muhmmad Siddique et al. [6] introduced the concept of S-topological group. In this paper, we initiate the study of generalized S topological group.***

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1. Introduction

In paper[2], Csaszar introduced the notions of generalized neighbourhood systems and generalized topological spaces. He also introduced the notions of associated interior and closure operators and continuous mappings on generalized neighborhood systems and generalized topological spaces. Csaszar investigated characterizations of generalized continuous mappings in [2]. The authors [6] introduced the concept of S topological group and some basic properties are investigated. The notion of generalized topological groups is studied by Murad Hussain and elt.all in[5]. A generalized topological group has the algebraic structure of groups and the topological structure of generalized topo- logical space.

In this paper, the notion of generalized S topological group is introduced. Generalized S topological groups are the groups with generalized topologies, multiplication and inversion being generalized semi continuous.

2. Preliminaries

The following definitions and results that are due to the authors [2], [5] and [6] will be useful in sequel.

Definition 2.1. [2] Let X be any set and let $\mathcal{G} \subset \mathcal{P}(X)$ be a subfamily of powerset of X. Then $\mathcal{G}$ is called a *generalized topology* if, $\varphi \in \mathcal{G}$ and for any index set I, $\cup_{i\in I} O_i \in \mathcal{G}$ whenever $O_i \in \mathcal{G}, i \in I$.

Generalized topology will be denoted by $\mathcal{G}-$topology.

Definition 2.2. [2] The elements of $\mathcal{G}$ are called $\mathcal{G}-$*open sets*. Similarly, a generalized closed set, or $\mathcal{G}-$closed set, is defined as complement of a $\mathcal{G}-$*open set*.

Definition 2.3. [2] Let X and Y be two $\mathcal{G}-$topological spaces. A mapping $f : X \to Y$ is called $\mathcal{G}-$*continuous* on X if for any $\mathcal{G}-$open set O in Y, $f^{-1}(O)$ is $\mathcal{G}-$open in X.

Definition 2.4. [2] The bijective mapping f is called a $\mathcal{G}-$homeomorphism from X to Y if both f and f^{-1} are $\mathcal{G}-$continuous. If there is a $\mathcal{G}-$homeomorphism between X and Y they are said to be $\mathcal{G}-$*homeomorphic* denoted by $X \cong_{\mathcal{G}} Y$.

Definition 2.5. [2] Collection of all $\mathcal{G}-$interior points of $A \subset X$ is called $\mathcal{G}-$interior of A. It is denoted by $Int_{\mathcal{G}}(A)$. By definiton it obvious that $Int_{\mathcal{G}}(A) \subset A$. Actually $\mathcal{G}-$interior of A, $Int_{\mathcal{G}}(A)$. is equal to union of all $\mathcal{G}-$open sets contained in A. Similarly we can define $\mathcal{G}-$*closure* of A as intersection of all $\mathcal{G}-$closed sets containing A. It is denoted by $Cl_{\mathcal{G}}(A)$. By definition $\mathcal{G}-$interior of A is a $\mathcal{G}-$open set while $\mathcal{G}-$closed sets is a $\mathcal{G}-$closed set.

Definition 2.6. [4] A subset A of a topological space X is said to be *semi-open* if there exists an open set U in X such that $U \subset A \subset Cl(U)$, or equivalently if $A \subset Cl(Int(A))$. The complement of a semi open set is said to be *semi-closed*.

Definition 2.7. [6] Let X and Y be topological spaces. A mapping $f : X \to Y$ is semi- continuous if for each open set V in Y, $f^{-1}(V) \in SO(X)$.

Definition 2.8. [6] A triple $(G, *, \tau)$ is said to be an *S-topological group* if $(G, *)$ is a group,
(G, τ) is a topological space, and
(a). the multiplication mapping $m : G \times G \to G$ defined by $m(x, y) = x * y, x, y \in G$, is semi-continuous.
(b). the inverse mapping $i : G \to G$ defined by $i(x) = x^{-1}, x \in G$, is semi continuous.

Definition 2.9. [5] $\mathcal{G}-$*topological group* is a group which is also a $\mathcal{G}$-topological space such that the multiplication map of G × G into G sending $x \times y$ into $x.y$, and the map of G into G sending x into x^{-1}, are $\mathcal{G}-$continuous maps.

Definition 2.10. [5] Let X be a set and $\Gamma \subset \mathcal{P}(X)$. We say that Γ is a covering of X if $X = \cup_{\gamma\in\Gamma}\gamma$. If X is a $\mathcal{G}-$topological space and every element of γ is $\mathcal{G}-$open(or $\mathcal{G}-$closed) then Γ is called $\mathcal{G}-$open covering(respectively $\mathcal{G}-$closed covering).

3. $\mathcal{G}-$S Topological Group

In this section interior and closure of a $\mathcal{G}-$ *S topological group* is denoted by $Int_{\mathcal{G}-S}$ and $Cl_{\mathcal{G}-S}$.

Definition 3.1. $\mathcal{G}-$S *topological group* is a group which is also a $\mathcal{G}-$topological space such that the multiplication map of G × G into G sending $x \times y$ into $x.y$, and the map of G into G sending x into x^{-1}, are $\mathcal{G}-$semi continuous maps.

Example 3.2. Let $G = Z_2 = \{0, 1\}$ be the two element(cyclic) group with the multiplica- tion mapping $m = +_2$ the usual addition modulo 2. Equip G with the Sierpinski topology $\tau = \{\varphi, \{0\}, G\}$. here

$SO(G \times G) = \{\varphi, \{(0, 0)\}, \{(0, 0), (0, 1)\}, \{(0, 0), (1, 0)\}, \{(0, 0), (0, 1), (1, 0)\}, \{(0, 0), (0, 1), (1, 0), (1, 1)\}, \{(0, 0), (1, 1)\}, \{(0, 0), (0, 1), (1, 1)\}, \{(0, 0), (1, 0), (1, 1)\}\}$.

and that $m : G \times G \to G$ is continuous at (0, 0), (1, 0), (0, 1), but not continuous at (1, 1). However, m is semi continuous at (1, 1). For this, let take us take the open set $V = \{0\}$ in G containing $m(1, 1) = 0$. Then the semi open set $U = \{(0, 0), (1, 1)\} \subset G \times G$ contains (1, 1) and $m(U) \subset V$. Therefore which is an $\mathcal{G}-$S topological group but not a topological group.

Proposition 3.3. *Any subgroup* H *of a* $\mathcal{G}-$S *topological group is a* $\mathcal{G}-$S *topological group again, called* $\mathcal{G}-$S *topological subgroup group of G.* Let us define morphisms of $\mathcal{G}-$S topological groups

Definition 3.4. Let $\varphi : G \longrightarrow G'$ be a mapping. Then φ is called a *morphism* of $\mathcal{G}-$S *topological groups.* If φ is both $\mathcal{G}-$ semicontinuous and group homomorphism.

φ is said to be $\mathcal{G}-$S topological group isomorphism if it is $\mathcal{G}-$S homeomorphism and group homomorphism.

Note 3.5. If we have a $\mathcal{G}-$S isomorphism between two $\mathcal{G}-$S topological groups G and G'

Definition 3.6. A $\mathcal{G}_{-S}$ topological space is said to be $\mathcal{G}_{-S}$ homogeneous if for any $x, y \in X$, there is a $\mathcal{G}-$S homeomorphism $\varphi : X \to X$ such that $\varphi(x) = y$.

Theorem 3.7. *Let* G *be* $\mathcal{G}-$S *topological group and* $\mathcal{G}-$S *Then the left(right) translation map* $L_g\,(R_g) : G \to G$, *defined by* $L_g(x) = gx (R_g(x) = gx)$, *is a* $\mathcal{G}-$S *topological homeomorphism.*

Proof. Here we will prove that L_g is a $\mathcal{G}-$S homeomorphism; similarly it can be shown that R_g is a $\mathcal{G}-$S homeomorphism. Let us prove that $L_g : G \to G$, is a $\mathcal{G}-$semi continuous. since $L_g : G \to G$ is equal to the composition

$G \xrightarrow{i_g} G \times G \xrightarrow{m} G,$

where $i_g(x) = (g, x), x \in G$ and m is the multiplication map in G then L_g is $\mathcal{G}-$semi continuous because i_g and m are $\mathcal{G}-$semi continuous. Here we should verify that the map $i_g : G \to G \times G$ is $\mathcal{G}-$semi continuous. For any $\mathcal{G}-$open set U × V, where U, V are $\mathcal{G}-$semi open sets in G,

$$i_g^{-1}(U \times V) = \begin{cases} V, \text{if } g \in U, \\ \varphi, \text{if } g \notin U, \end{cases}$$

Since $\mathcal{G}-$semi open set in the product $\mathcal{G}-$topology of $G \times G$ can be written as a union of $\mathcal{G}-$semi open sets of the form U×V. Then i_g is $\mathcal{G}-$semicontinuous. Since $(L_g)^{-1} = L_{g^{-1}}$ is $\mathcal{G}-$ semicontinuous, the left translation map $L_g : G \times G$ is a $\mathcal{G}-$S homeomorphism.

Theorem 3.8. *Let G be a $\mathcal{G}-S$ topological group, U a $\mathcal{G}-$semi open subset of G, and A be any subset of G. Then the set AU(respectively, UA) is $\mathcal{G}-$semi open in G.*

Theorem 3.9. *Let G be a $\mathcal{G}-S$ topological group. Then for every subset A of G and every $\mathcal{G}-$ semi open set U containing the identity element e, $Cl_{\mathcal{G}-S}$ (A) $\subset$ AU ($Cl_{\mathcal{G}-S}$ (A) $\subset$ UA).*

Proof. Since the inversion is $\mathcal{G}-$semi continuous, we can find a $\mathcal{G}-$semiopen set V containing e such that $V^{-1} \subset U$. Take $x \in Cl_{\mathcal{G}-S}$. Then xV is a $\mathcal{G}-$semi open set containing x, therefore there is $a \in A \cap xV$, that is $a = xb$, for some $b \in V$. Then $x = ab^{-1} \in AV^{-1} \subset AU$, hence, $Cl_{\mathcal{G}-S}(A) \subset AU$.

Theorem 3.10. *Let G be a $\mathcal{G}-S$ topological group, and $\mathcal{B}_e$ a base of the space G at the identity element e. Then every subset A of G,*

$$Cl_{\mathcal{G}-S}(A) = \bigcap\{AU : U \in \mathcal{B}_e\}$$

Proof. In view of Theorem 3.8, we only verify that if x is not in $Cl_{\mathcal{G}-S}(A)$, then there exists $U \in \mathcal{B}_e$ such that $x \notin AU$. Since $x \notin Cl_{\mathcal{G}-S}(A)$, there exists a $\mathcal{G}-$semi open neighbourhood W of e such that $(xW) \cap A = \Phi$ which is obviously implies that AU does not contain x.

Theorem 3.11. *Let $f : G \to H$ be $\mathcal{G}-S$ morphism. If f is $\mathcal{G}-$semi continuous at the identity e_G of G, then f is $\mathcal{G}-$semi continuous at every $g \in G$.*

Proof. Let $g \in G$ be any point. Suppose that O is a $\mathcal{G}-$semi open neighbourhood of $h = f(g)$ in H. Since left translation L_h is a $\mathcal{G}-S$ homeomorphism of H, there exists a $\mathcal{G}-$semi open neighbourhood U of e_G in G such that $f(U) \subset V$. Since L_g is a $\mathcal{G}-S$ homeomorphism of G onto itself, the set gU is a $\mathcal{G}-$semi open neighbourhood of g in G, and we have that $f(gU) = hf(U) \subset hV \subset O$. Hence f is $\mathcal{G}-$semi continuous at the point g.

Theorem 3.12. *Let G be a $\mathcal{G}-S$ topological group and let H be a subgroup of G. If H contains a non-empty $\mathcal{G}-$ semi open set, then H is $\mathcal{G}-$semi open in G.*

Proof. Let U be a non-empty $\mathcal{G}-$semi open subset of G with $U \subset H$. For every $h \in H$, $L_h(U) = hU$. Therefore, the subgroup $H = \cup_{h \in H} hU$ is $\mathcal{G}-$semi open in G by $Uh \subset H, \forall h \in H$.

Definition 3.13. Let X be a set and $\Gamma \subset \mathcal{P}(X)$. We say that Γ is a covering of X if $X = \cup_{\gamma \in \Gamma} \gamma$. If X is a $\mathcal{G}-S$ topological space and every element of γ is $\mathcal{G}-$semi open(or $\mathcal{G}-$semi closed) then Γ is called $\mathcal{G}-$semi open covering(respectively $\mathcal{G}-$semi closed covering).

If $\mathcal{G}-S$ topological space has a $\mathcal{G}-$semi open covering then it must be strong.

Theorem 3.14. *Let G be a $\mathcal{G}-S$ topological group and let H be a subgroup of G. If H is a $\mathcal{G}-$semi open set, then it is also $\mathcal{G}-$semi closed in G.*

Proof. Let $\Gamma = \{gH : g \in G\}$ be the family of all left cosets of H in G. This family is a disjoint $\mathcal{G}-$semi open covering of G by left translations. Therefore, every element of Γ, being the complement of the union of all other elements, is $\mathcal{G}-$semi closed in G. In particular, $H = eH$ is $\mathcal{G}-$semi closed in G.

Theorem 3.15. *G be a $\mathcal{G}-S$ topological group. Then $\mathcal{G}-$semi closure of any subgroup of G is a $\mathcal{G}-S$ topological subgroup again.*

Proof. Let H be a subgroup of G. First we prove that $Cl_{\mathcal{G}-S}(H)$ is closed under multiplication m in G. Given $x, y \in Cl_{\mathcal{G}-S}(H)$ and for any $\mathcal{G}-$semi open set U containing xy, we need to show that $U \cap H = \varphi$. Since $m : G \times G \to G$ is $\mathcal{G}-$ semi continuous, there exist $\mathcal{G}-$semi open sets V and W containing x and y, respectively, such that $m(V \times W) \subset U$. Since $x, y \in Cl_{\mathcal{G}-S}(H)$ then we have $V \cap H = \varphi$ and $W \cap H = \varphi$

Hence $\varphi \neq m(V \times W) \cap H \subset U \cap H$ which implies that $xy \in Cl_{\mathcal{G}-S}((H)$.

Now $Cl_{\mathcal{G}-S}(H)$ is closed under the inverse operation because $(Cl_{\mathcal{G}-S}(H))^{-1} \subset Cl_{\mathcal{G}-S}(H^{-1}) = Cl_{\mathcal{G}-S}(H)$.

Theorem 3.16. *Let G be a $\mathcal{G}-S$ topological group. Then $\mathcal{G}-$semi closure of any invariant subgroup of G is a $\mathcal{G}-S$ topological invariant subgroup again.*

Proof. Suppose H is an invariant subgroup in G. By above theorem, $Cl_{\mathcal{G}-S}(H)$ is a subgroup of G. Now we prove that $Cl_{\mathcal{G}-s}(H)$ is invariant. Given $g \in G$, let $k_g : G \to G$ be conjugation by g, i.e. $k_g(h) = ghg^{-1} = L_g o R_{g^{-1}}(h)$. Then k_g is a $\mathcal{G}$-S homeomorphism from G to itself. By lemma1.7 $k_g(Cl_{\mathcal{G}-s}(H)) \subset Cl_{\mathcal{G}-s}(k_g(H)) = Cl_{\mathcal{G}-s}(H)$, $\forall g \in G$.

It implies that
$k_g(Cl_{\mathcal{G}-s}(H)) = g Cl_{\mathcal{G}-s}(H) g^{-1}$
$\subset Cl_{\mathcal{G}-s}(H)$, $\forall g \in G$.

Hence $Cl_{\mathcal{G}-s}(H)$ is an invariant subgroup of G.

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