Conditions of Boundedness for Compact operators on Banach spaces

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Abstract: In this paper we study Compact operator between Banach spaces which are bounded. We establish two conditions that must be satisfied for boundedness of Compact operator.

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1. Introduction

A Banach space is a normed linear space that a complete metric space with respect to the metric derived from its norm. See [7]. This space has been studied by several authors in different context such a in [3] , [1] and [6] . We give a brief discussion of conditions for boundedness of Compact operator. First Condition is the existence of the constant M ≥ 0 as we done in Remark 3 that the operator T:X → Y between two normed operator bounded by letting T is compact which associate with the closures of the images of B(0,2) and B(0, n) producing \( \|T\| = \sup_{x \in B(0,2)} \|Tx\| \leq \sup_{x \in B(0,2)} \|Tx\| \) . Since \( x \in B(0,2) \) ∈ \( \mathbb{R} \) we use a constant M instead of \( \sup_{x \in B(0,2)} \) for some \( M \in \mathbb{R} \), hence \( \|Tx\| \leq M \) is bounded as long as \( TB(0,2) \subset T(B(0,2)) \) and \( T(B(0,2)) \) is compact and hence bounded. In Lemma 4 we put \( S = \{x \in x: \|x\| = 1\} \). Since T(S) is compact and \( \|x\| = 1 \) this result \( \|T\| \leq M \) which is bounded by constant M. Second condition is the operator T must be continuous at every point. See Theorem 6 we show that the operator T:X → Y is bounded by letting T is continuous for all \( x, y \in X \) we have \( \|Tx - Ty\| = \|T(x - y)\| \leq M \|x - y\| \) where M is a constant, then by letting T is continuous at 0. Since T is linear we have \( T(0) = 0 \) there exists \( \delta > 0 \) with \( \|x\| \leq \delta \) and \( x \neq 0 \) we define \( \delta \) by \( \delta = \frac{\|x\|}{\|\|\|} \). Then \( \|x\| \leq \delta \). From linearity of T that \( \|Tx\| = \|\frac{\|x\|}{\|\|} \|T\|\|x\| \leq M \|x\| \) where \( M = \frac{1}{\delta} \). Thus T is bounded and continuous at zero, then at every point. (See Remark 3, Lemma 4, Definition 5, and Theorem 6).

2. Compact operators on Banach space

A continuous linear operator T : X → Y on Banach spaces is compact when T maps bounded sets in X pre-compact sets in Y , that is, sets with compact closure. Since bounded sets lies in some ball in X , and since T is linear , its suffices to verify that T maps the unit ball in X to a pre-compact set in Y . Finite –rank operators are clearly compact. Both right and left compositions ToS and STo of compact T with continuous S produce compact operators. Recall a criterion for pre-compactness: a set E in a complete metric space is pre-compact if and only if its totally bounded , sense that , given \( \varepsilon > 0 \) , E is covered by finitely-many open balls of radius \( \varepsilon \). We claim that operator-norm limits \( T = \lim_{i \to \infty} T_i \) of compact operators \( T_i \) are compact : given \( \varepsilon > 0 \) , choose \( T_i \) so that \( \|T_i - T\|_o \leq \varepsilon \), cover the image of the Unit ball \( B_1 \) under \( T_i \) by finitely-many open ball \( U_k \) of radius \( \varepsilon \).

Since \( \|T_i x - T x\| < \varepsilon \) for all \( x \in X \) , enlarging the balls \( U_k \) to radius \( 2\varepsilon \) covers \( TB_1 \).

(If desired , rewrite the proof replacing \( \varepsilon \) by \( \frac{\varepsilon}{2} \) ) . So there exist Banach spaces with compact operators which are not norm-limits of finite –rank operators . Recall that every compact operators T : X → Y on Hilbert space is an operator norm limit of finite rank operators . Given \( \varepsilon > 0 \) , let \( y_1 , ..., y_n \) be the centers of open \( \varepsilon – \) balls in Y covering \( TB_1 \), where \( B_1 \) is the unit ball in X . Let \( T_\varepsilon = P O T \) where P is the projection of Y to the span of the \( y_i \), then similarly , the sum of two compact operator is compact.

Definition 1

A compact operator is a linear operator T from Banach space X to another Banach space Y , such that the image under T of any bounded subset of X is necessarily a bounded operator , and so continuous . Any bounded operator Y that has finite rank is a compact operator , the class of compact operators is a natural generalization of the class of finite –rank operators in an infinite-dimensional setting . When Y is a Hilbert space , it is true that any compact is a limit of finite-rank operators can be defined alternatively as the closure in the operator norm of the finite –rank operators.

Lemma 2 : Let K be a compact operator on \( \mathcal{H} \) and suppose \( (T_n) \) is a bounded sequence in \( B(\mathcal{H}) \) such that , for each x ∈ \( \mathcal{H} \) the sequence \( (T_n x) \) converges to \( T x \) , where \( T \in B(\mathcal{H}) \). Then \( (T_n K) \) converges to TK in norm. Briefly, the above can be rephrased as: If \( K \in \mathcal{K}(\mathcal{H}) \) and \( \|T_n x - T x\| \to 0 \) for all \( x \in \mathcal{H} \) then \( \|T_n K - T K\| \to 0 \).

In words: Multiplying by a compact operator on the right converts a pointwise convergent sequence of operators into a norm convergent one.

Proof

Since \( (T_n) \) is bounded sequence , \( \|T_n\| \leq M \) for some constant M . Then for all \( x \in \mathcal{H} \), \( \|T x\| = \lim_{n \to \infty} \|T_n x\| \leq M \|x\| \) and so \( \|T\| \leq M \).

Let K be compact and suppose that \( \|TK - T_n K\| \to 0 \). Then there exists some \( \delta > 0 \).
And a subsequence \((T_n, K)\) such that \(\|TK - T_nK\| > \delta\).

Choose unit vectors \(x_{n_j}\) of \(X\) such that \(\|(T - T_n)Kx_{n_j}\| > \delta\). That this can be done follows directly from the definition of the norm of an operator. Using the fact that \(K\) is compact, we can find a subsequence \((x_{n_j})\) of \((x_{n_i})\) such that \((Kx_{n_j})\) is convergent. Let the limit of this sequence by \(x\). Then for all \(\delta > 0\), there exists \(n\) so that \(\|Kx_{n_j} - y\| < \frac{\delta}{M}\).

Now, using the convergence of \((Kx_{n_j})\) to \(y\), there exists \(n\) so that, for \(n_j > n\),

\[
\|Kx_{n_j} - y\| < \frac{\delta}{M}.
\]

Also, using the convergence of \((T_n)\) to \(T\), there exists \(m\) so that, for \(n_j > m\),

\[
\|(T - T_n)y\| < \frac{\delta}{M}.
\]

Then, for \(j > \max\{n, m\}\) that right hand side of the displayed inequality is less than \(\frac{\delta}{4}\), and this contradiction shows that the supposition that \(\|TK - T_nK\| \rightarrow 0\) is false.

3. First Condition of Boundedness of Compact operator

Remark 3: A compact operator is bounded.

Proof

First we show that a compact linear operator \(T : X \rightarrow Y\) between normed spaces is bounded, if \(T\) is compact, then the closure of the image of \(B(0,1)\) and \(B(0, n)\) is compact, therefore, \(\|T\| = \sup_{x \in B(0,1)} \|T(x)\| \leq M\).

Lemma 4: Every compact operator is bounded.

Proof

Put \(S = \{x \in X : \|x\| = 1\}\), then \(T(S)\) is relatively compact, hence bounded (by \(M\)).

Definition 5

Let \(X\) and \(Y\) be two normed linear spaces. We denote that both \(X\) and \(Y\) norms by \(\|\cdot\|\). A linear operator \(T : X \rightarrow Y\) is bounded if there is constant \(M \geq 0\) such that \(\|T(x)\| \leq M\|x\|\) for all \(x \in X\).

4. Second Condition of Boundedness of Compact operator

Theorem 6

Linear compact operator is bounded if it is continuous.

Proof

First, suppose that \(T : X \rightarrow Y\) is bounded. Then, for all \(x, y \in X\), we have

\[
\|Tx - Ty\| = \|T(x - y)\| \leq M\|x - y\|.
\]

Where \(M\) is a constant for which \((1)\) holds, therefore we can take \(\epsilon = M\).

In the definition of continuity, \(T\) is continuous. Second, suppose that \(T\) is continuous at 0. Since \(T\) is linear, we have \(T(0) = 0\).

Choosing \(\epsilon = 1\) in the definition of continuity, we conclude that there is a \(\delta > 0\) such that \(\|Tx\| \leq 1\) whenever \(\|x\| \leq \delta\) for any \(x \in X\), with \(x \neq 0\).

Then \(\|x\| \leq \delta\), so \(\|Tx\| \leq 1\).

It follow from linearity of \(T\) that \(\|Tx\| = \frac{\|x\|}{\delta} \|Tx\| \leq M\|x\|\).

Where \(M = \frac{1}{\delta}\). Thus \(T\) is bounded.

References


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