On Ternary Quadratic Homogeneous Diophantine Equation $5x^2 + 5y^2 – 9xy = 20z^2$

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Abstract- ***Five different methods of the non-zero integral solutions of the ternary quadratic homogeneous Diophantine equation $5x^2 + 5y^2 – 9xy = 20z^2$ are obtained. Some interesting relations among the special numbers and the solutions are exposed.***

Keywords: Ternary Quadratic, homogenous cone, integer solutions, special numbers.

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Notations Used

$t_{m,n}$ – Polygonal number of rank n with size m
P_n – Pronic number of rank n
g_a – Gnomonic number of rank a
P_n^m - Pyramidal number of rank n with size m

1. Introduction

The number theory is queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [3-12]. In this work, we are observed another interesting five different methods of the non-zero integral solutions the ternary quadratic homogeneous Diophantine equation $5x^2 + 5y^2 – 9xy = 20z^2$. Further, some elegant properties among the special numbers and the solutions are observed.

2. Method of Analysis

The ternary quadratic homogeneous Diophantine equation to be solved is

$5x^2 + 5y^2 – 9xy = 20z^2$ (1)

Introducing the linear transformation

$x = u + v, y = u – v, \quad u \neq v \neq 0$ (2)

in (1), it reduces to

$u^2 + 19v^2 = 20z^2$ (3)

We discuss below five different methods of non-zero distinct integer solutions of (3)

2.1 Method: 1

Let $z = a^2 + 19b^2$, where $a, b \neq 0$ (4)

Write 20 as

$20 = (1 + i\sqrt{19}) (1 - i\sqrt{19})$ (5)

Substituting (4) & (5) in (3) and employing the method of factorization, we get

$(u + i\sqrt{19} v) = (1 +i\sqrt{19}) (a + i\sqrt{19} b)^2$

On equating real and imaginary parts, we get

$u = u (a, b) = a^2 – 19b^2 – 38 ab$
$v = v (a, b) = a^2 – 19b^2 + 2ab$

In view of (2), we get the non-zero integral solutions of (1) in two parameters.

$x = x (a, b) = 2a^2 – 38b^2 – 36ab$
$y = y (a, b) = - 40 ab$

Properties

1. $x (A, A) + 74 t_{4, A} = 0$
2. $x (A, A + 1) + 72 t_{4, A} + g_{56A} + 39 = 0$
3. $y (A, A + 1) + 40P_A = 0$
4. $y (A, A) + 40 t_{4, A} = 0$
5. $z (A, B) – t_{4, A} – 19t_{4, B} = 0$
6. $z (A(A + 1), B(B + 1)) - P_A^2 - 19P_B^2 = 0$
7. $y (A(A+1), A + 80 P_A^5 = 0$

2.1 Method: 2

Write (3) as

$19v^2 = 20z^2 – u^2$ (6)

Write 19 as

$19 = (\sqrt{20} + 1)(\sqrt{20} - 1)$ (7)

Let $v = 20 a^2 – b^2$, where $a, b > 0$ (8)

Using (7) in (6), we get

$20z^2 – u^2 = (\sqrt{20} + 1)(\sqrt{20} - 1)(20a^2 - b^2)^2$

On employing the method of factorization & equating rational & irrational parts, we get

$z = z (a, b) = 20a^2 + b^2 + 2ab$
$u = u (a, b) =20a^2 + b^2 + 40ab$

Substituting u & v in (2), we get the non-zero integral solutions of (1) in two parameters

$z = z (a, b) = 40a^2 + 40ab$
$y (a, b) = 2b^2 + 40ab$

Properties

1. $x (A, A+ 1) – 40 t_{4,A} – 40P_A = 0$
2. $y (A(A+1), 1) – 40P_A = 2$
3. $z (B, B+1)-23t_{4, B} – g_{2B} = 0$
4. $x (A(A+1),1) – 40 P_A^2 - 40P_A = 0$
5. $y (A,1)-2t_{44, A} + 42 t_{4,A} \equiv 0 \pmod 2$
6. $x (A,A(A+1))-40t_{4, A} – 80 P_A^5 =0$
7. $z (A,1)-2t_{22, A} + g_{8A} = 0.$

2.3 Method: 3

Write (3) as

$1*u^2 = 20z^2 - 19v^2$ (9)

Assume $u = u(a,b) = 20a^2 - 19b^2$

$= (\sqrt{20}a + \sqrt{19}b)(\sqrt{20}a - \sqrt{19}b)$ (10)

Write 1 as

$1 = (\sqrt{20} + \sqrt{19})(\sqrt{20} - \sqrt{19})$ (11)

Using (10) & (11) in (9), we get

$20z^2 - 19v^2 =$

$(\sqrt{20}) + \sqrt{19})$

$(\sqrt{20} - \sqrt{19})(\sqrt{20}a + \sqrt{19}b)^2 (\sqrt{20}a - \sqrt{19}b)^2$

On employing the method of factorization and equating the rational & irrational parts, we get

$z = z(a, b) = 20a^2 + 19b^2 + 38ab$

$v = v(a, b) = -20a^2 + 19b^2 + 40ab$

Substituting u & v in (2), we get

$x = x(a, b) = 40a^2 + 40ab$

$y = y(a, b) = -38b^2 - 40ab$

Properties

1. $x(A, A + 1) - 40\, t_{4,A} - 40\, P_A = 0$
2. $y(A, 1) + g_{20A} + 39 = 0$
3. $z(A, A) - 77\, t_{4,A} = 0$
4. $z(1, B) - t_{4,B} - g_{19B} \equiv 1 \pmod 2$
5. $x(A, A(A+1)) - 40t_{4,A} - 80\, P_A^5 = 0$
6. $y(A, A) + 2\, t_{80,A} - g_{38A} = 1$
7. $z(A(A+1), 1) - 20\, P_A^2 - 30\, P_A = 19$

2.4 Method: 4

Write (3) as

$u^2 + 19v^2 = 19z^2 + z^2$

$u^2 - z^2 = 19z^2 - 19v^2$

$(u + z)(u - z) = 19(z + v)(z - v)$ (12)

Write (12) as

$\frac{u+z}{z-v} = \frac{19(z+v)}{u-z} = \frac{A}{B}, B \neq 0,$

which is equivalent to the system of equations in three unknowns

$Bu + Z(B - A) + AV = 0$

$-Au + Z(A + 19B) + 19BV = 0$

By cross- multiplication, we get

$u = u(A, B) = -A^2 + 19B^2 - 38AB$

$v = v(A, B) = -A^2 + 19B^2 + 2AB$

$z = z(A, B) = -A^2 - 19B^2$

Substituting the values of u & v in (2), we get

$x = x(A, B) = -2A^2 + 38B^2 - 36AB$

$y = y(A, B) = -40AB$

$z = z(A, B) = -A^2 - 19B^2$

Properties

1. $x(A, 1) + 2t_{40,A} - g_{18A} \equiv 0 \pmod 3$
2. $z(A, A) + 2t_{22,A} - g_{9A} = 1$
3. $y(A, A+1) + 40\, P_A = 0.$
4. $x(A(A+1), 1) + 2\, P_A^2 + 36P_A \equiv 0 \pmod 2$
5. $z(A, B) + t_{4,A} + 19t_{4,B} = 0$
6. $y(A, A(A+1)) + 80\, P_A^5 = 0$
7. $x(A, A(A+1)) + 2t_{4,A} - 38\, P_A^2 + 72\, P_A^5 = 0$

2.5 Method: 5

Write 20 as

$20 = \frac{(31 + i\sqrt{19})(31 - i\sqrt{19})}{49}$ (13)

Assume $z = a^2 + 19b^2$ (14)

Substituting equation (13) & (14) in (3) and employing the method of factorization, we get

$(u + i\sqrt{19}\, v) = \frac{1}{7}(9 + i\sqrt{19})(9 + i\sqrt{19}b)^2$

Equating real and imaginary parts, we get

$u = u(a, b) = \frac{1}{7}(31a^2 - 5891b^2 - 38ab)$

$v = v(a, b) = \frac{1}{7}(a^2 - 19b^2 + 62ab)$

In view of (2), we get

$x = x(a, b) = \frac{1}{7}[32a^2 - 608b^2 + 24ab]$ (15)

$y = y(a, b) = \frac{1}{7}(30a^2 - 570b^2 - 100ab)$ (16)

Choose a, b suitably so that x, y, and z are in integers. Putting a = 7A, b = 7B in (15) & (16), we get the non-zero integral solutions of (1) in two parameters

$x = x(A, B) = 224A^2 - 4256B^2 + 168AB$

$y = y(A, B) = 210A^2 - 3990B^2 - 700AB$

$z = z(A, B) = 49A^2 + 931B^2$

Properties

1. $z(A, B) - 49\, t_{4,A} - 931\, t_{4,B} = 0$
2. $z(A(A+1), B)) - 49\, P_A^2 - 931\, t_{4,B} = 0$
3. $y(1, A(A+1)) - 3990\, P_A^2 + 700\, P_A \equiv 0 \pmod 7$
4. $x(A, A) + 3864\, t_{4,A} = 0$
5. $x(A(A+1), B(B+1)) - 49P_A^2 - 931\, P_B^2 = 0$
6. $y(1,B) + 3990\, t_{4,B} + g_{350B} \equiv 0 \pmod 3.$
7. $z(1, A(A+1)) - 931P_A^2 \equiv 0 \pmod 7$
8. $x(B(B+1), 1) - 224\, P_B^2 - 168P_B \equiv 0 \pmod 2$

3. Conclusion

In this work, we have observed five different patterns of the non-zero integral solutions of the ternary quadratic homogeneous Diophantine equation $5x^2 + 5y^2 - 9xy = 20z^2$ and relations between solutions and special numbers are also obtained. One may research for any other patterns of this equation and their corresponding properties.

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