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Integral Solutions of the Homogeneous Biquadratic Diophantine Equations with Five Unknowns $(X^2 - Y^2)(3X^2 + 3Y^2 - 2XY) = 12(Z^2 - W^2)T^2$

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Abstract: ***Four different patterns are used to find non-zero distinct integral solutions for the homogeneous biquadratic Diophantine equations $(X^2 - Y^2)(3X^2 + 3Y^2 - 2XY) = 12(Z^2 - W^2)T^2$. Different types of properties are exposed in every pattern with polygonal, nasty, square and cubic numbers.***

Keywords: Homogeneous biquadratic, integral solutions, special numbers

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Notations used

$T_{m,n}$ - Polygonal number of rank n with size m
P_n^m - Pyramidal number of rank n with size m
g_n - Gnomonic number of rank n
Pr_n - Pronic number of rank n
CP_n^6 - Centered hexagonal pyramidal number of rank n
OH_n - Octahedral number of rank n
SO_n - Stella octangular number of rank n

1. Introduction

The number theory is queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [3-12]. In this work, we are observed another interesting four different methods of the non-zero integral solutions of the homogeneous Biquadratic Diophantine equations with five unknowns $(X^2 - Y^2)(3X^2 + 3Y^2 - 2XY) = 12(Z^2 - W^2)T^2$. Further, some elegant properties among the special numbers and the solutions are observed.

2. Method of Analysis

The homogeneous Biquadratic Diophantine equations to be solved is

$$(X^2 - Y^2)(3X^2 + 3Y^2 - 2XY) = 12(Z^2 - W^2)T^2 \qquad (1)$$

Introducing the linear transformations

$$X = u + v,\ Y = u - v,\ Z = 2u + v,\ W = 2u - v, \text{ where } u \neq v \neq 0 \qquad (2)$$

in (1), we get

$$u^2 + 2v^2 = 6T^2 \qquad (3)$$

Assume $T(a, b) = a^2 + 2b^2$, where $(a, b \neq 0)$ (4)

We present below four different patterns of non-zero distinct integer solutions to (1).

2.1 Pattern 1:

Take 6 as

$$6 = (2 + i\sqrt{2})(2 - i\sqrt{2}) \qquad (5)$$

Substituting (4) and (5) in (3) and applying the method of factorization, we get

$$(u + i\sqrt{2}v) = (2 + i\sqrt{2})(a + i\sqrt{2}b)^2$$

Equating real and imaginary parts, we get

$$u = 2a^2 - 4b^2 - 4ab \qquad (6)$$
$$v = a^2 - 2b^2 + 4ab \qquad (7)$$

Putting (6) and (7) in (2), we get non-zero distinct integer valued for x, y, z, w and satisfying (1) are given below

$$X = X(a, b) = 3a^2 - 6b^2 \qquad (8)$$
$$Y = Y(a, b) = a^2 - 2b^2 - 8ab \qquad (9)$$
$$Z = Z(a, b) = 5a^2 - 10b^2 - 4ab \qquad (10)$$
$$W = W(a, b) = 3a^2 - 6b^2 - 12ab \qquad (11)$$

The equations (8) to (11) and (4) give non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. $X(A, A(A+1)) + 24T_{3,A}^2 = 3T_{4,A}$
2. $Y(A(A+1), A+2) - 4T_{3,A}^2 + T_{6,A} + 48P_A^3 \equiv -8 \pmod 9$
3. $X(A, A+1) - 3Y(A, A+1) = 8Pr_A$
4. $5Y(A, 1) - Z(A, 1) + g_{15A} + 1 = 0$
5. $8T(A, A)$ is a Nasty number
6. $-6Y(A, A)$ is a Nasty number
7. $-2\{w(A, A)\}$ is a Nasty number

2.2 Pattern:2

Instead of (5), Take 6 as

$$6 = (-2 + i\sqrt{2})(-2 - i\sqrt{2}) \qquad (12)$$

We do the same procedure as in pattern:1 and from (2) we get

$$X = X(a, b) = -a^2 + 2b^2 - 8ab \qquad (13)$$
$$Y = Y(a, b) = -3a^2 + 6b^2 \qquad (14)$$
$$Z = Z(a, b) = -3a^2 + 6b^2 - 12ab \qquad (15)$$

W=W (a, b) = $-5a^2 + 10b^2 - 4ab$ (16)

The equations (13) to (16) and (4) give non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. $X(A, A(A+1)) + T_{4,A} + 16P_A^5 = 8T_{3,A}^2$
2. $X(A, 1) - 3Y(A, 1) + T(A, 1) - T_{4,A} \equiv 2 \pmod{24}$
3. $5Y(A, 1) - Z(A, 1) + T(A, 1) - T_{4,A} \equiv 2 \pmod{36}$
4. $X(A, 1) - W(A, 1) + T(A, 1) - 12Pr_A \equiv 2 \pmod{11}$
5. $\{W(A, A)\}$ is a Square number

Each of the following represents a Nasty number

6. $8\{Y(A, A)\}$
7. $\frac{8}{3}\{Z(A, A)\}$
8. $-12\{X(A, A)\}$

2.3 Pattern:3

We write 6 as

$$6 = \frac{(22 + i\sqrt{2})(22 - i\sqrt{2})}{81} \quad (17)$$

Using (17) and (4) in (3), and applying method of factorization, we get

$$(u + i\sqrt{2}v) = \frac{(22 + i\sqrt{2})}{9}(a + i\sqrt{2}b)^2 \quad (18)$$

Equating real and imaginary part, we get

$$u = \frac{1}{9}[22a^2 - 44b^2 - 4ab]$$

$$v = \frac{1}{9}[a^2 - 2b^2 + 44ab]$$

In view of (2), we get the values of X, Y, Z, and W as

$$X = X(a, b) = \frac{1}{9}[23a^2 - 46b^2 + 40ab] \quad (19)$$

$$Y = Y(a, b) = \frac{1}{9}[21a^2 - 42b^2 - 48ab] \quad (20)$$

$$Z = Z(a, b) = \frac{1}{9}[45a^2 - 90b^2 + 36ab] \quad (21)$$

$$W = W(a, b) = \frac{1}{9}[43a^2 - 86b^2 - 52ab] \quad (22)$$

Choose a and b so that the values of X, Y, Z, and W are in integer,

Putting a = 3A, b = 3B in (19) to (22) and (4), we get

X = X (a, b) = $23a^2 - 46b^2 + 40ab$ (23)

Y =Y (a, b) = $21a^2 - 42b^2 - 48ab$ (24)

Z = Z (a, b) = $45a^2 - 90b^2 + 36ab$ (25)

W=W (a, b) = $43a^2 - 86b^2 - 52ab$ (26)

T=T (a, b) = $9A^2 + 18B^2$ (27)

The equations (23) to (27) give non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. $9X(A, A+1) - 23T(A, A+1) + T_{1658,A} - 36Pr_A \equiv -828 \pmod{2483}$
2. $Z(A, A(A+1)) - 5T(A, A(A+1)) = 72P_A^5$
3. $Y(1, B) + T_{86,B} \equiv 21 \pmod{89}$
4. $X(A, A^2) + 40CP_n^6 + 69T_{4,A}^2 = 23Pr_{A^2}$
5. $W(A, 2A^2+1) + 344Pr_{A^2} + 43T_{4,A} + 156OH_A \equiv 0 \pmod 2$
6. $Z(A, 2A^2-1) + 360Pr_{A^2} - 45T_{4,A} + 36SO_A \equiv 0 \pmod 5$
7. $-\frac{1}{13}\{X(A, A) + W(A, A)\}$ is a Nasty number
8. $-\frac{3}{13}\{Z(A, A) + W(A, A)\}$ is a Nasty number

2.4 Pattern:4

Rewrite (1) as

$$6T^2 - u^2 = 2v^2 \quad (28)$$

Consider 2 as

$$2 = (\sqrt{6} + 2)(\sqrt{6} - 2) \quad (29)$$

Assume v (a, b) = $6a^2 - b^2$; $a, b \neq 0$ (30)

Substituting (29), (30) in (28) and applying method of factorization, we get

$$(\sqrt{6}T + u) = (\sqrt{6} + 2)(\sqrt{6}a + b)^2 \quad (31)$$

Equating real and imaginary part, we get

T = T (a, b) = $6a^2 + b^2 + 4ab$ (32)

u = u (a, b) = $12a^2 + 2b^2 + 12ab$ (33)

Applying (30) and (33) in (2), we get

X = X (a, b) = $18a^2 - b^2 + 12ab$ (34)

Y =Y (a, b) = $6a^2 + 3b^2 + 12ab$ (34)

Z = Z (a, b) = $30a^2 + 3b^2 + 24ab$ (35)

W=W (a, b) = $18a^2 + 5b^2 + 24ab$ (36)

T=T (a, b) = $6a^2 + b^2 + 4ab$ (37)

The equation (34) to (37) give the non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. $X(A, A+1) - T_{36,A} - g_{7A} = 12Pr_A$
2. $5Y(1, B) - Z(1, B) - 12Pr_B - g_{12B} = 1$
3. $X(1, B) - W(1, B) + T_{14,B} \equiv 0 \pmod{19}$
4. $Y(B, B) - T(B, B) - T_{6,B} \equiv 0 \pmod 3$
5. $\frac{1}{2}\{X(A, A) + Y(A,A)\}$ is a Square number
6. $\{Y(A, A^2) - T(A, A^2) - 2T_{4,A}^2\}$ is a Cubic number
7. $\frac{3}{2}\{Y(A,A) + T(A, A)\}$ is a Nasty number

3. Conclusion

In this work, we have observed four different patterns of the non-zero integer solutions of the homogeneous Biquadratic Diophantine equation $(X^2 - Y^2)(3X^2 + 3Y^2 - 2XY) = 12(Z^2 - W^2)T^2$ and relations between solutions and special numbers are also obtained. One may research for any other patterns of this equation and their corresponding properties.

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