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Dominating-χ-Color Number of Harary Graph

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Abstract: ***Let $G = (V, E)$ be a graph. A graph G is k-colorable if it has a proper k-coloring. The chromatic number $\chi(G)$ is the least k such that G is k- colorable. In this paper, we examine the relation between the domination number, chromatic number and dominating χ-color number of Harary graph $H_{k,n}$. And we answer a question posed in [1] by showing that if q = 0, then $\chi(H_{2r,n}) = p$ if not p + 1 such that $n = (r + 1)s + t$ and $n = ps + q$ with $r \geq t \geq 0, s > q \geq 0$.***

Keywords: Proper coloring, Chromatic number, Maximal independent set, Domination number, Dominatingχ-color number.

1. Introduction

Let $G = (V, E)$ be a simple, connected, finite, undirected graph. The order and size of G are denoted by n and m respectively [1].

In graph theory, coloring and domination are two important areas which have been extensively studied. The fundamental parameter in the theory of graph coloring is the chromatic number $\chi(G)$ of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color [3,4].

The minimum cardinality of a maximal independent set of a graph G is called the lower independence number and is denoted by i(G). A set $D \subseteq V$ is a dominating set of G, if for every vertex $x \in V \setminus D$ there is a vertex $y \in D$ with $xy \in E$. The minimum cardinality of a dominating set of G is the domination number and is denoted by $\gamma(G)$. Since a set is maximal independent if and only if it is a dominating set, $\gamma(G) \leq i(G)$ [5].

For a given k with $2 \leq k \leq n$, the Harary graph $H_{k,n}$ is constructed as follows:
Place n vertices $v_1, v_2, \ldots\ldots, v_n$
i) If k is even. Let k = 2r, then every vertex v_i is adjacent to the consecutive r vertices on either side of v_i.
ii) If k is odd and n is even. Let k = 2r + 1 and n = 2m, then every vertex is adjacent with 2r vertices as previously mentioned and then adding edges joining vertex v_i to v_{i+m}
iii) If k is odd and n is odd .Let k = 2r + 1 and n = 2m - 1, then $H_{k,n}$ is obtained from $H_{k-1,n}$ by adding the edge $v_i v_{i+m}$ for each $1 \leq i \leq \frac{n-1}{2}$ and $v_1 v_m$. It is clear that $H_{2,n} = c_n$ and $H_{n-1,n} = K_n$. Further $\chi(H_{2,2n}) = 2, \chi(H_{2,2n+1}) = 3, \chi(H_{n-1,n}) = n$.

2. Preliminary Results

Let G be a graph of order n whose n vertices are listed in some specified order. In greedy coloring, if the vertices of G are listed in the order $v_1, v_2, \ldots\ldots, v_n$, then the resulting greedy coloring assigns the color 1 to v_1. If v_2 is not adjacent to v_1, then assign the color 1 to v_2, otherwise assign the color 2. In general, suppose first j vertices $v_1, v_2, \ldots\ldots, v_j$, $1 \leq j < n$ in the sequence have been colored and t is the smallest positive integer not used in any neighbor of v_{j+1} from among $v_1, v_2, \ldots\ldots, v_j$. Then assign the color t to v_{j+1} [6].

Definition 2.1 [2]

Let G be a graph with $\chi(G) = k$. Let $C = \{V_1, V_2, \ldots\ldots, V_k\}$ be a k-coloring of G. Let d_c denote the number of color classes in C which are dominating sets of G. Then $d_\chi(G) = \max_c d_c$ where the maximum is taken over all the χ- colorings of G, is called the dominating -χ-color number of G. Arumugam et al. [2] prove the following theorems,

Theorem 2.2 [2]
If G is uniquely χ- colorable, then $d_\chi(G) = \chi(G)$

Proposition 2.3 [2]
For $n \geq 3$, $d_\chi(C_n) = \begin{cases} 3, \text{if } n \equiv 3 (\text{mod} 6) \\ 2, \text{otherwise} \end{cases}$

The following observations are straight forward,
1. For all graphs G, $1 \leq d_\chi(G) \leq \chi(G)$
2. $\chi(G) = n$ iff $d_\chi(G) = n$ iff $G = K_n$
3. If G is bipartite, then $d_\chi(G) = 2$.
4. For all graphs G, $d_\chi(G) \leq \delta(G) + 1$.

Theorem 2.4 (Brooks' Theorem)
For any graph G, $\chi(G) \leq \Delta(G)$, unless G is a complete graph or an odd cycle.

Theorem 2.5 [6]
If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$

Theorem 2.6 [6]
For every graph G, $\omega(G) \leq \chi(G)$.

3. Chromatic Number of $H_{k,n}$

Let $H_{k,n} = H_{2r,n}$ or $H_{2r+1,n}$ be a Harary graph with n vertices $v_1, v_2, \ldots\ldots, v_n$ as defined in the introduction.

Lemma 3.1
Let $H_{k,n} = H_{2r,n}$ or $H_{2r+1,n}$, with $r \geq 2$. Then

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$$\chi(H) \geq \begin{cases} r+1 \text{ if } r+1 \backslash n \\ r+2 \text{ otherwise} \end{cases}$$

Proof:

Case - 1: If $r + 1$divide n.

Define the coloring function, c : V $\rightarrow \{1,2,\ldots.,r+1\}$ defined by $c(v_i) = j$ with$i \equiv j(\text{mod}\, r+1)$ and $1 \leq j \leq r+1$. Also every $r+1$ consecutive vertices of H has a $r + 1$- clique. By Theorem 2.6, the graph H has $r + 1$- clique, and therefore $\chi(H) \geq r + 1$.

Case - 2: If r + 1 does not divides n.

Coloring can be made in such a way that, Divide the vertex into $\{V_1, V_2, \ldots\ldots, V_p\}$ with r + 1 vertices each and with r + 1 different colors. Since r + 1 does not divides n, the remaining $\{v_1, v_2, \ldots\ldots, v_i\}$ vertices with $1 \leq i < r + 1$ and $i < p$ will be colored r + 2 and it occupies each vertex set increasing the color class from r + 1 to r + 2. Thus $\chi(H) \geq r + 2$.

Theorem 3.2

For every $n = p\left\lfloor\frac{n}{r+1}\right\rfloor + q$ with $0 \leq q \leq \left\lfloor\frac{n}{r+1}\right\rfloor$, we have

$$\chi(H_{2r,n}) = \begin{cases} p \text{ if } q = 0 \\ p+1 \text{ otherwise} \end{cases}$$

Proof:

Case - 1: If $q = 0$. Clearly, $r + 1$ divides n and $n = p\left\lfloor\frac{n}{r+1}\right\rfloor$. Therefore, by lemma $\chi(H_{2r,n}) = p$.

Case - 2: If $q \neq 0$. Clearly, $r + 1$ does not divides n and $n = p\left\lfloor\frac{n}{r+1}\right\rfloor + q$. By assuming $q = 0$ color the $p\left\lfloor\frac{n}{r+1}\right\rfloor$vertices with p colors and the remaining$\{v_1, v_2, \ldots\ldots, v_i\}$, $i < p$ vertices will be given a color $p+1$ and it occupies each vertex set increasing the color class from p to $p+1$.Therefore $\chi(H_{2r,n}) = p+1$. For $r = 1, H_{2r,n} = C_n$. Since $n = 2s + t$ with the conditions $n = ps + q$with $r \geq t \geq 0, s > q \geq 0$, we find for all $n, p = 2$. If n is even, then $q = 0$ that is $\chi(C_n) = 2$ and if n is odd, then $q \neq 0$ that is $\chi(C_n) = 3$.

The interesting question which is first appeared in print in [1] is

Question [1]

If $r \geq 3$, then can one say $\chi(H_{2r,3r+2}) \neq r+2$?

Solution:

Here, $n = 3r + 2$, implies that $n = (r + 1)2 + r$. If r is an even number, then n can be rewritten as $(r+1+$ $r22+0$ and if r is an odd number, then$r + 1 + r-122 + 1$. By theorem 3.2, the chromatic number of $H_{2r,3r+2}$ is either$\left(r+1+\frac{r}{2}\right)2$ or $\left(r + 1 + \frac{r-1}{2}\right)$ and therefore $\chi(H_{2r,3r+2}) = r+2+\left\lfloor\frac{r}{2}\right\rfloor$. Since$\left\lfloor\frac{r}{2}\right\rfloor > 0$, it is clear that $\chi(H_{2r,3r+2}) > r+2$

Theorem 3.3

Let G = $H_{2r+1,2m}$, then

$$\chi(H_{2r+1,2m}) = \begin{cases} \chi(H_{2r,2m}) \text{ if } r+1 \nmid m \\ \chi(H_{2r,2m})+1 \text{ otherwise} \end{cases}$$ with$m-1 > r \geq 1$.

Proof:

Since $H_{2r+1,2m}$ is obtained from adding the edges $v_i v_{i+m}$ with $1 \leq i \leq n$ in the graph $H_{2r,2m}$. By Theorem 2.5, the graph $H_{2r,2m}$is the subgraph of $H_{2r+1,2m}$ and therefore$\chi(H_{2r,2m}) \leq \chi(H_{2r+1,2m})$. If $r+1 \nmid m$, then the increment of edges $v_i v_{i+m}$ does not affect the coloring. The colors of $H_{2r,2m}$ is enough to get the proper coloring of$H_{2r+1,2m}$, so$\chi(H_{2r+1,2m}) = \chi(H_{2r,2m})$. If not, $\chi(H_{2r,2m})\backslash$ m. So, v_1 and v_m are in the color class V_1 and v_m is adjacent with atleast one vertex in each color class. Therefore v_m has a color other then 1,2,3, ... , $\chi(H_{2r,2m})$.

There is a result analogous to Theorem 3.3 that expresses the chromatic number of a graph $H_{2r+1,2m}$ in terms of the chromatic number of $H_{2r,2m}$.

For r = 1; $\chi(H_{2r+1,2m})$ = 3 and $r = m-1$; $\chi(H_{2r+1,2m})$ = 2m.

Theorem 3.4

Let G = $H_{2r+1,2m-1}$, then $\chi(H_{2r+1,2m-1})$

$$= \begin{cases} \chi(H_{2r,2m-1}) \text{ if } \chi(H_{2r,2m-1}) \nmid m \text{ or } m-1 \\ \chi(H_{2r,2m-1})+1 \text{ otherwise} \end{cases}$$

with $m-1 > r \geq 1$.

Proof:

Since $H_{2r+1,2m-1}$ is obtained from adding the edges $v_i v_{i+m}$ with $1 \leq i \leq n$ in the graph $H_{2r,2m-1}$. By Theorem 2.5, the graph $H_{2r,2m-1}$is the subgraph of $H_{2r+1,2m-1}$ and therefore$\chi(H_{2r,2m} - 1) \leq \chi(H_{2r+1,2m-1})$. If $\chi(H_{2r,2m-1}) \nmid$ m or m-1, then the increment of edges$v_i v_{i+m}$ does not affect the coloring. The colors of $H_{2r,2m-1}$ is enough to get the proper coloring of$H_{2r+1,2m-1}$, so$\chi(H_{2r+1,2m-1}) = \chi(H2r,2m-1)$. If not, $\chi(H2r,2m-1)\backslash$m or m$-$1. Assume v_1 is adjacent with v_m (or v_{m-1}). And also v_m and v_{m-1} are adjacent with atleast one vertex in each color class. Therefore v_m (or v_{m-1}) has a color other then 1,2,3,…… , $\chi(H_{2r,2m-1})$.

There is a result analogous to Theorem 3.4 that expresses the chromatic number of a graph $H_{2r+1,2m-1}$ in terms of the chromatic number of $H_{2r,2m-1}$. For r = 1; $\chi(H_{2r+1,2m-1})$ = 3 and $r = m - 2$; $\chi(H_{2r+1,2m-1}) = m$.

To sharpen the lower bound of chromatic number in the lemma 3.1, we derive the following Theorem 3.5 from Theorems 3.2, 3.3 and 3.4.

Theorem 3.5

For any graph $H_{k,n} = H_{2r,n}$ or $H_{2r+1,n}$,with $r \geq 2$, then $\chi(H_{k,n}) \geq p$ such that $n = (r + 1)s + t$ and $n = ps + q$ with $r \geq t \geq 0$; $s > q \geq 0$.

Proof:

Since s is cardinality of maximal independent set of vertices of $H_{k,n}$, $\chi(H_{k,n}) \geq \frac{n}{s}$.Thus$\chi(H_{k,n}) \geq p$.

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4. The relation between χ, γ and d_χ

In this section we derive the relation between the parameters χ, γ and d_χ of the Harary graph $H_{k,n}$.

Theorem 4.1

For any graph $H_{2r,n}$, we have $d_\chi(H_{2r,n}) = p$ such that $n = (r + 1)s + t$ and $n = ps + q$ with $r \geq t \geq 0$; $s > q \geq 0$.

Proof:

Let $V_i, 1 \leq i \leq k$ be the color class of $H_{2r,n}$. If $q = 0$, then $H_{2r,n}$ is uniquely p-colorable. By Theorem 2.2, $H_{2r,n}$ is uniquely p-colorable and therefore $d_\chi(H_{2r,n}) = \chi(H_{2r,n})) = p$. If $q \neq 0$, then $\chi(H_{2r,n}) = p + 1$. We claim that the color class V_{p+1} with q vertices is not a dominating set. Every v_i dominates v_j, with $i - r \leq j \leq i + r$ (where addition is taken to modulo n). There is no adjacent vertices v_j , with $n - r + 1 \leq j \leq n$ of v_1 has the color p+1 because $r(< p)$ is not a multiple of p. Since no vertex of V_{p+1} dominates v_1, dominating-χ-color number is $d_\chi(H_{2r,n}) = p$.

The following is then an immediate consequence of the Theorem 4.1. It gives the relation between the dominating -χ- color number and chromatic number of a Hararygraph $H_{2r,n}$.

Corollary 4.2

Let $n = (r + 1) s + t$ and $n = ps + q$ with $r \geq t \geq 0$; $s > q \geq 0$.

$$\chi(H_{2r,n}) = \begin{cases} d_\chi(H_{2r,n}) \text{ if } q = 0 \\ d_\chi(H_{2r,n}) + 1 \text{ otherwise} \end{cases}$$

Theorem 4.3

For any Harary graph, $H_{k,n} = H_{2r,n}$ or $H_{2r+1,n}$, with $r \geq 1$, then $1 \leq \gamma(H_{k,n}) \leq s$ such that $n = (r + 1) s + t$ and $n = ps + q$ with $r \geq t \geq 0$; $s > q \geq 0$.

Proof:

An obvious lower bound on the domination number is one. Since s is cardinality of maximal independent set, the domination number of $H_{k,n}$ is less than or equal to s.

The following result offers the relation between domination number, chromatic number and dominating -χ-color number of a Harary graph $H_{k,n}$ [7,8].

Theorem 4.4

For any Harary graph $H_{k,n}$ with $k \geq 2$, $d_\chi \leq \frac{n}{\gamma} \leq \chi + 1$

Proof:

Let c be the cardinality of the color class V_i with $1 \leq i \leq d_\chi$, which can dominate all other vertices of a graph. By definition of dominating χ color number, $c < \frac{n}{d_\chi}$

First, to show that $\gamma d_\chi < n$, assume to the contrary, that $\gamma d_\chi > n$. Then $c < \frac{n}{d_\chi} < \gamma$. Therefore the number of c with $c < \gamma$, vertices can dominate the graph. This is contradiction.

To show that $n \leq \gamma(\chi + 1)$, assume to the contrary, that $n > \gamma(\chi + 1)$. And let s as defined in the Theorem 4.3. Then $n/(\chi+1) > \gamma$. Since $\leq r + 1, s > \gamma$. By Theorem 4.3, which contradicts the fact that $< \gamma$.

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