Solving Hybrid Fuzzy Fractional Differential Equations by Runge Kutta 4th Order Method

S. Ruban Raj[1], M. Saradha[2]

[1]Associate Professor, Department of Mathematics, St. Joseph College, Trichirappalli, Tamilnadu, India

[2]Research Scholar, Department of Mathematics, St. Joseph College, Trichirappalli, Tamilnadu, India

Abstract: *In this paper we study numerical methods for hybrid fuzzy fractional differential equation, Degree of Sub element hood and the iteration method is used to solve the hybrid fuzzy fractional differential equations with a fuzzy initial condition.*

Keywords: hybrid fuzzy fractional differential equations, Degree of Sub Element Hood

Mathematical Subject Classification (MSC) Code: 65L06

1. Introduction

Fuzzy initial value problems for fractional differential equations have been considered by some authors recently [2, 3]. To study some dynamical processes, it is necessary to take into account imprecision, randomness or uncertainty. The objective of the present paper is to extend the application of the iteration method, to provide approximate solutions for fuzzy initial value problems of differential equations of fractional order, and to make comparison with that obtained by an exact fuzzy solution.

2. Analytical solution of Hybrid Fuzzy Fractional Differential Equations

Let us consider the following fractional differential equation

$$_c D_a^\beta x(t) = f(t, x(t), \lambda_k(x_k)), \; t \in [t_k, t_{k+1}] \qquad (1)$$

$$x(t_k) = x_k$$

Where, $0 \le t_0 \le t_1 \le \cdots \le t_k \to \infty$

3. The Fourth Order Runge Kutta Method With Harmonic Mean

For a hybrid fuzzy fractional differential equation we develop the fourth order Runge Kutta method with harmonic mean when f and λ_k in (1) can be obtained via the Zadeh extension principle from:

$f \in [R^+ \; X \; R \; X \; R, \; R]$ and $\lambda_k \in C\,[R,R]$

we assume that the existence and uniqueness of solutions of (1) hold for each $[t_k, t_{k+1}]$. For a fixed r, to integrate the system in (3) $[t_0, t_1], [t_1, t_2], \ldots [t_k, t_{k+1}] \ldots$ we replace each interval by a set of N_{k+1} discrete equally spaced grid points (including the end points) at which the exact solution $x(t; r) = (\underline{x}(t;r), \overline{x}(t;r))$ is approximated by some $(\underline{y}(t;r), \overline{y}(t;r))$. For the chosen grid points on $[t_k, t_{k+1}]$ at

$$t_{k,n} = t_k + n h_k, \; h_k = \frac{t_{k+1} - t_k}{N_k}, \; 0 \le n \le N_k.$$

Let $(\underline{Y}_k(t;r), \overline{Y}_k(t;r) \equiv (\underline{x}_k(t;r), \overline{x_k}(t;r))$, $(\underline{y}_k(t;r), \overline{y}_k(t;r))$ and $(\underline{y}_k(t;r), \overline{y}_k(t;r))$ may be denoted respectively by $(\underline{Y}_{k,n}(t;r), \overline{Y}_{k,n}(t;r))$ and $(\underline{y}_{k,n}(t;r), \overline{y}_{k,n}(t;r))$.

We allow N_k's to vary over the $[t_k, t_{k+1}]$'s so that the h_k's may be comparable. The Fourth Order Runge Kutta method for (1) is given by:

$(\underline{Y}_k(t;r), \overline{Y}_k(t;r) \equiv (\underline{x}_k(t;r), \overline{x_k}(t;r))$,

$(\underline{y}_k(t;r), \overline{y}_k(t;r))$

Where

$$\underline{k_1}(t_{k,n}; y_{k,n}(r)) = \min \left\{ \begin{array}{l} h_k f(t_{k,n}, u, \lambda_k(u_k)) \\ \backslash u \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)], \\ u_k \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)] \end{array} \right\},$$

$$\overline{k_1}(t_{k,n}; y_{k,n}(r)) = \max \left\{ \begin{array}{l} h_k f(t_{k,n}, u, \lambda_k(u_k)) \\ \backslash u \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)], \\ u_k \in [\underline{y}_{k,n}(r), \overline{y}_{k,n}(r)] \end{array} \right\},$$

$$\underline{k_2}(t_{k,n}; y_{k,n}(r)) = \min \left\{ \begin{array}{l} h_k f(t_{k,n} + \frac{1}{2}(h_k), u, \lambda_k(u_k)) \\ \backslash u \in \left[\begin{array}{c} \underline{\Phi}_{k_1}(t_{k,n}, y_{k,n}) \\ \overline{\Phi}_{k_1}(t_{k,n}, y_{k,n}) \end{array} \right] \\ u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \end{array} \right\},$$

$$\underline{k}_2(t_{k,n}; y_{k,n}(r)) = \max \left\{ \begin{array}{l} h_k f(t_{k,n} + \frac{1}{2}(h_k), u, \lambda_k(u_k)) \\ \backslash u \in \left[\dfrac{\underline{\Phi}_{k_1}(t_{k,n}, y_{k,n})}{\overline{\Phi}_{k_1}(t_{k,n}, y_{k,n})} \right] \\ u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \end{array} \right\},$$

$$\underline{k}_4(t_{k,n}; y_{k,n}(r)) = \min \left\{ \begin{array}{l} h_k f(t_{k,n} + h_k, u, \lambda_k(u_k)) \\ \backslash u \in \left[\dfrac{\underline{\Phi}_{k_3}(t_{k,n}, y_{k,n})}{\overline{\Phi}_{k_3}(t_{k,n}, y_{k,n})} \right] \\ u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \end{array} \right\},$$

$$\overline{k}_4(t_{k,n}; y_{k,n}(r)) = \max \left\{ \begin{array}{l} h_k f(t_{k,n} + h_k, u, \lambda_k(u_k)) \\ \backslash u \in \left[\dfrac{\underline{\Phi}_{k_3}(t_{k,n}, y_{k,n})}{\overline{\Phi}_{k_3}(t_{k,n}, y_{k,n})} \right] \\ u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \end{array} \right\}$$

$$\underline{k}_3(t_{k,n}; y_{k,n}(r)) = \min \left\{ \begin{array}{l} h_k f(t_{k,n} + \frac{1}{2}(h_k), u, \lambda_k(u_k)) \\ \backslash u \in \left[\dfrac{\underline{\Phi}_{k_2}(t_{k,n}, y_{k,n})}{\overline{\Phi}_{k_2}(t_{k,n}, y_{k,n})} \right] \\ u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \end{array} \right\},$$

Where

$$\underline{\Phi}_{k_1}(t_{k,n}, y_{k,n}(r)) = \underline{y}_{k,n}(r) + \frac{1}{2}\underline{k}_1(t_{k,n}, y_{k,n}(r))$$

$$\overline{\Phi}_{k_1}(t_{k,n}, y_{k,n}(r)) = \overline{y}_{k,n}(r) + \frac{1}{2}\overline{k}_1(t_{k,n}, y_{k,n}(r))$$

$$\underline{\Phi}_{k_2}(t_{k,n}, y_{k,n}(r)) = \underline{y}_{k,n}(r) + \frac{1}{2}\underline{k}_2(t_{k,n}, y_{k,n}(r))$$

$$\overline{k}_3(t_{k,n}; y_{k,n}(r)) = \max \left\{ \begin{array}{l} h_k f(t_{k,n} + \frac{1}{2}(h_k), u, \lambda_k(u_k)) \\ \backslash u \in \left[\dfrac{\underline{\Phi}_{k_2}(t_{k,n}, y_{k,n})}{\overline{\Phi}_{k_2}(t_{k,n}, y_{k,n})} \right] \\ u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \end{array} \right\},$$

$$\overline{\Phi}_{k_2}(t_{k,n}, y_{k,n}(r)) = \overline{y}_{k,n}(r) + \frac{1}{2}\overline{k}_2(t_{k,n}, y_{k,n}(r))$$

$$\underline{\Phi}_{k_3}(t_{k,n}, y_{k,n}(r)) = \underline{y}_{k,n}(r) + \underline{k}_3(t_{k,n}, y_{k,n}(r))$$

$$\overline{\Phi}_{k_3}(t_{k,n}, y_{k,n}(r)) = \overline{y}_{k,n}(r) + \overline{k}_3(t_{k,n}, y_{k,n}(r))$$

Next we define:

$$S_k[t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)]$$

$$= \frac{1}{6}\{\underline{k}_1(t_{k,n}; y_{k,n}(r)) + 2[\underline{k}_2(t_{k,n}; y_{k,n}(r)) + \underline{k}_3(t_{k,n}; y_{k,n}(r))] + \underline{k}_4(t_{k,n}; y_{k,n}(r))\}$$

$$T_k[t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)]$$

$$= \frac{1}{6}\{\overline{k}_1(t_{k,n}; y_{k,n}(r)) + 2[\overline{k}_2(t_{k,n}; y_{k,n}(r)) + \overline{k}_3(t_{k,n}; y_{k,n}(r))] + \overline{k}_4(t_{k,n}; y_{k,n}(r))\}$$

The exact solution at $t_{k,n+1}$ is given by:

$$\begin{cases} F_{k,n+1}(r) = \underline{Y}_{k,n}(r) + S_k[t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)], \\ G_{k,n+1}(r) = \overline{Y}_{k,n}(r) + T_k[t_{k,n}, \underline{y}_{k,n}(r), \overline{y}_{k,n}(r)]. \end{cases}$$

………(4)

Where $\left| F_{k,n+1} \right| = \sum_{e_j \in A} \exp(F_{k,n+1})$

and

$$\text{\$}(G_{k,n+1}, F_{k,n+1}) = \frac{1}{\left|(G_{k,n+1})\right|}\{\left|(G_{k,n+1})\right| - \sum \max\{0, (G_{k,n+1}) - (F_{k,n+1})\}\}.$$

4. Degree of Sub Element hood

Let X be a Universal, U be a set of parameters and let $(F_{k,n+1})$ and $(G_{k,n+1})$ are two fuzzy elements of X. Then the degree of sub element hood denoted by $\text{\$}(F_{k,n+1}, G_{k,n+1})$ is defined as,

$$\text{\$}(F_{k,n+1}, G_{k,n+1}) = \frac{1}{\left|(F_{k,n+1})\right|}\{\left|(F_{k,n+1})\right| - \sum \max\{0, (F_{k,n+1}) - (G_{k,n+1})\}\}$$

5. Numerical Examples

*In this section, we present the examples for solvi*ng hybrid fuzzy fractional differential equations.

Example: 1
Consider the following linear hybrid fuzzy fractional differential equation:

$$_c D_a^\beta X(t) = 1 + X^2 \qquad \qquad \dots\dots (5)$$
$$X(0) = X_0,$$

where $\beta \in (0,1]$, t > 0, and

X_0 is any triangular fuzzy number.

This problem is a generalization of the following hybrid fuzzy fractional differential equation:

$$_c D_a^{\beta} \, x(t) = 1 + x^2(t) = 1 + [\underline{x}(t;r), \overline{x}(t;r)]^2 \qquad \dots(6)$$

$$x(t) = x_0.$$

Where $\beta \in (0,1]$,

t > 0, α is the step size and x0 is a real number.

We can find the solution of the hybrid fractional fuzzy differential equation, by the method of Runge kutta 4th order Method. We compared & generalized the hybrid fractional fuzzy differential equation solution with the exact solution in the following table; also we illustrated the figure for this generalization by using Matlab.

Table 1: Numerical Solution of Example 1

α	$Y_{k,n+1}$	$F_{k,n+1}$	$G_{k,n+1}$
0.1	1.187298247867520	0.787298247867520	.587298247867520
0.2	1.362772659265217	0.962772659265217	1.762772659265216
0.3	1.533606860771214	1.133606860771214	1.933606860771214
0.4	1.702940459085539	1.302940459085540	2.102940459085539
0.5	1.872820986069762	1.472820986069763	2.272820986069762
0.6	2.044858431782411	1.644858431782411	2.444858431782411
0.7	2.220494863603316	1.820494863603317	2.620494863603316
0.8	2.401149486106885	2.001149486106885	2.801149486106885
0.9	2.588315276211985	2.188315276211985	2.988315276211985
1.0	2.7836636789711434	2.383636789711434	3.183636789711434

$|F_{k,n+1}|$ = 15.69789406

$|G_{k,n+1}|$ = 23.69789406

$\mathcal{S}(F_{k,n+1}, G_{k,n+1}) \cong 1$

$\mathcal{S}(G_{k,n+1}, F_{k,n+1})$ = 0.66241726 $\cong$ 0.66
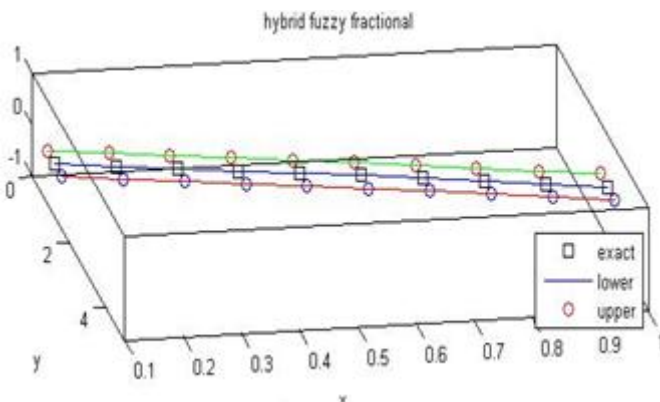
Figure 1: Comparison of exact and approximated solution of Example 1

6. Conclusion

In this paper, we have studied a hybrid fuzzy fractional differential equation. Final results showed that the solution of hybrid fuzzy fractional differential equations approaches the solution of hybrid fuzzy differential equations as the fractional order approaches the integer order. The results of the study reveal that the proposed method with fuzzy fractional derivatives is efficient, accurate, and convenient for solving the hybrid fuzzy fractional differential equations.

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