A New Method for Construction of MV-optimal Generalized Group Divisible Designs with two Groups

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Abstract: ***In this article, we consider the construction of generalized group divisible designs with two groups (GGDD (2)) from balanced incomplete block designs (BIBD). We also discuss MV-optimality of these designs.***

Keywords: Balanced Incomplete Block Design (BIBD); Group Divisible Design (GDD); Generalized Group Divisible Design (GGDD); MV-optimality

1. Introduction

Consider a design with v treatments arranged in b blocks of size k each. Let $D(v,b,k)$ denote the class of all such designs which are also connected. In an experimental set up, the choice of a design is usually determined by some optimality criteria. Among a number of optimality criteria, the one called MV-optimality is of great importance. The MV-optimality criterion was introduced by Takeuchi [5]. An MV-optimal design minimizes the maximum variance over all paired treatment contrasts among all the designs in $D(v,b,k)$. Jacroux [3] has described some innovative methods of construction of MV-optimal generalized group divisible designs. Thannippara et al. [1] discussed the construction of MV-optimal generalized group divisible designs. Srivastav and Morgan [4] considered MV-optimality of GGDD(s). In this article we introduce a relatively easy method of construction of MV-optimal Generalized Group Divisible Designs with two groups (GGDD(2)). GGDD(2) are relatively easy to construct and relatively simple to analyze. We found that the generalized group divisible designs obtained by Thannippara et al. [1] are also MV-optimal.

2. Preliminaries

2.1 Definitions

Balanced Incomplete Block Designs (BIBD): An incomplete block design with v treatments distributed over b blocks each of size k, ($k<v$) such that each treatment occurs in r blocks, no treatment occurs more than once in a block and each pair of treatments occurs together in λ blocks, is called a balanced incomplete block design. The symbols $v,\ b,\ r,\ k,$ and λ are the parameters of the design

Generalized Group Divisible Design with s Groups (GGDD(s)): Let $d(v,b,k)$ be a design having v treatments arranged in b blocks of size k. Then d is called a generalized group divisible design with s groups if the treatments in d can be divided into s mutually disjoint sets $V_1,V_2,\cdots,V_s$ of size $v_1,\ v_2,\cdots,v_s$ such that

1. For $i=1,2,\cdots,s$ and for all $a\in V_i$, $r_{da}=\lambda_{da}=r_i$, where r_i is a constant.
2. For $i,j=1,2,\cdots,s$ and for all $a\in V_i$, $b\in V_j$, $a\neq b$, $\lambda_{ab}=\lambda_{ij}$ where λ_{ij} depends only on the treatments V_i and V_j.

MV-optimality: A design d is said to be MV-optimal in the class $D(v,b,k)$ if the maximal variance with which it estimates elementary treatment differences is minimal among all designs in the class $D(v,b,k)$.

Lemma 2.1. (Takeuchi, [5]). Let $d\in D(v,b,k)$ be an arbitrary design. Then for any i and j, $i\neq j$, the variance with which $\tau_i-\tau_j$ is estimated in d satisfies

$$\mathrm{Var}(\hat{\tau}_i-\hat{\tau}_j)\geq 4k/[(r_{d_i}+r_{d_j})(k-1)+2\lambda_{d_{ij}}].\ \square$$

Theorem 2.1. (Srivastav and Morgan, [4]). Let $D(v,b,k)$ be a class of designs such that $r(k-1)=\lambda(v-1)+q$ for some $0\leq p\leq v-1$ and $bk=vr+p$ for some $0\leq p\leq v-2$. Let $d^*\in D(v,d,k)$ be any GGDD(s) satisfying

1. $\gamma_{ss}\geq \mathrm{Int}[(pk+2r(k-1))/2(v-1)]$

2. $c_g \geq c$ for $1 \leq g \leq s-1$ and $c_s = c$

3. $\gamma_{gg} \geq \gamma_{ss}$ for $1 \leq g \leq s-1$

4. For $1 \leq g, h \leq s$ with $g \neq h$, $\gamma_{gh} = \gamma_{1s}$ i.e., γ_{gh} is constant in $g \neq h$

5. $\dfrac{\gamma_{gg}+(v-1)\gamma_1}{v\gamma_{1s}(kc_g+\gamma_{gg})} + \dfrac{\gamma_{hh}+(v-1)\gamma_{1s}}{v\gamma_{1s}(kc_h+\gamma_{hh})} \leq \dfrac{2}{kc+\gamma_{ss}}$

for $1 \leq g, h \leq s$ with $g \neq h$

Then d^* is MV-optimal in $D(v,b,k)$. □

3. Method of Construction

Consider a BIBD, say $d(v,\bar{b},k,r,\lambda)$. Suppose that we are deleting a block from this BIBD. The resulting design is a GGDD which has always two groups and hence GGDD(2). One of these groups has $(v-k)$ treatments and the other has k treatments. Let these groups be denoted by V_1 and V_2 respectively. Obviously the parameters of this GGDD(2) are v, $b = \bar{b}-1$, $r_1 = \bar{r}$, $r_2 = \bar{r}-1$, $\gamma_{11} = \lambda_{11}$, $\gamma_{22} = \lambda_{22} = \lambda - 1$, and $\gamma_{12} = \lambda_{12} = \lambda$.

4. Example

Consider a symmetrical BIBD with parameters $v = b = 4$, $r = k = 3$, and $\lambda = 2$. Let the blocks of this design be

1	1	1	2
2	2	3	3
3	4	4	4

Deleting the blocks $(2,3,4)$ from this BIBD, we get a new design, say, d^* whose blocks are

1	1	1
2	2	3
3	4	4

The resulting design d^* is a GGDD(2) $V_1 = (1)$ and $V_2 = (2,3,4)$.

5. Optimality

Theorem 5.1: Let $D(v,b,k)$ be a class of designs such that $bk = vr + p$, $0 < p < v$. Suppose $d^* \in D(v,b,k)$ is a GGDD(2) and satisfies:

(i) $\gamma_{22} \geq \mathrm{Int}\left[(p+2r-v)(k-1)/2(v-1)\right]$

(ii) $c_1 \geq c$ and $c_2 = c$

(iii) $\gamma_{11} \geq \gamma_{22}$

(iv) $\gamma_{ij} = [r(k-1)-(v-p-1)\gamma_{22}]/p$ for all $i \neq j$; $i, j = 1, 2$

If γ_{22} also satisfies the following conditions

$$\frac{1}{(r+1)(k-1)+\gamma_{22}} + \frac{1}{r(k-1)+\gamma_{12}} \leq \frac{2}{r(k-1)+\gamma_{22}}$$

and

$$k+1+\frac{2[r(k-1)-(v-1)(\gamma_{22}+1)]}{p} < 1$$

then d^* is MV-optimal in $D(v,b,k)$.

Proof. Consider the matrix $T_{d_x} = C_d + xJ_{vv}$, where C_d is the information matrix, J_{vv} is the $v \times v$ matrix of ones, x is any positive real number. The covariance matrix for estimates of treatment effect is given by $T_{d_x}^{-1}$. Now substituting $x = \gamma_{12}/k$ in above, we have

$$T_{d_x} = \left(\frac{1}{k}\right)\mathrm{diag}[(kc_i+\gamma_{ii})I_{v_i} + (\gamma_{1s}-\gamma_{ii})J_{v_i}], \quad i = 1, 2, \cdots, s$$

so that

$$T_{d_x}^{-1} = (k)\mathrm{diag}\left[\frac{1}{(kc_i+v_{ii})}I_{v_i} - \frac{\gamma_{1s}-\gamma_{ii}}{(kc_i+\gamma_{ii})[(kc_i+\gamma_{ii})+v_i(\gamma_{1s}-\gamma_{ii})]}J_{v_i}\right]$$

Since C matrix of design d satisfies the relationships

$$C_d \cdot \bar{1} = 0$$

$$kc_i + \gamma_{ii} + v_i(\gamma_{1s}-\gamma_{ii}) = v\gamma_{1s},$$

and thus

$$T_{d_x}^{-1} = (k)\mathrm{diag}\left[\frac{1}{(kc_i+v_{ii})}\left(I_{v_i} + \frac{\gamma_{ii}-\gamma_{1s}}{v\gamma_{1s}}J_{v_i}\right)\right].$$

Now for $i, j \in V_1$

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) \leq \frac{2k}{kc_1+\gamma_{22}} = \frac{2k}{kc+\gamma_{22}} = m^* \text{ (say), as}$$

$c \leq c_1$ and $\gamma_{22} \leq \gamma_{11}$.

For $i \in V_1$ and $j \in V_2$

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) = \frac{k}{kc_1+\gamma_{11}} + \frac{k}{kc_2+\gamma_{22}}$$

$$+\frac{k(\gamma_{11}-\gamma_{12})}{v\gamma_{12}(kc_1+\gamma_{11})} + \frac{k(\gamma_{22}-\gamma_{12})}{v\gamma_{12}(kc_2+\gamma_{22})}$$

$$= \frac{k(\gamma_{11}+(v-1)\gamma_{12})}{v\gamma_{12}(kc_1+\gamma_{11})} + \frac{k(\gamma_{22}+(v-1)\gamma_{12})}{v\gamma_{12}(kc_2+\gamma_{22})}$$

$$= \frac{k(\lambda+(v-1)\lambda)}{v\lambda(kc_1+\lambda)} + \frac{k(\lambda-1+(v-1)\lambda)}{v\lambda(kc_2+\lambda-1)}$$
$$\le \frac{kv\lambda}{v\lambda(kc_1+\lambda)} + \frac{kv\lambda}{v\lambda(kc_2+\lambda-1)}$$
$$\le \frac{2k}{kc+\gamma_{22}} = m^*,$$

Now for $i \in V_2$ and $j \in V_2$

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) \le \frac{2k}{kc+\gamma_{22}} = m^*.$$

Thus for all values of i and j between $1 \le i, j \le v$,

$$\mathrm{Var}\,(\hat{\tau}_i - \hat{\tau}_j) \le \frac{2k}{kc+\gamma_{22}}.$$

Now let $d \in D(v,b,k)$ and $r_{d_1} \ge r_{d_2} \ge \cdots \ge r_{d_v}$. If $r_{d_v} < r$, then

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) \ge \frac{4k}{(r_{d_i}+r_{d_v})(k-1)+2\lambda_{d_{iv}}} = \frac{4k}{A_{iv}},$$

where $A_{iv} = (r_{d_i}+r_{d_v})(k-1)+2\lambda_{d_{iv}}$.

Now the sum

$$\sum_{i=1}^{v-1} A_{iv} = (k-1)\left[(v-1)r_{d_v} + \sum_{i=1}^{v-1} r_{d_i}\right] + 2\sum_{i=1}^{v-1}\lambda_{d_{iv}}$$

$$= (k-1)\left[(v-1)r_{d_v} + (bk - r_{d_v})\right] + 2r_{d_v}(k-1)$$
$$= (k-1)[bk+(v-2)r_{d_v}] + 2r_{d_v}(k-1)$$
$$= bk(k-1) + vr_{d_v}(k-1)$$
$$\le (k-1)(vr+p) + v(k-1)(r-1)$$
$$= vkr + kp - vr - p + vkr - vk - vr + v$$
$$= 2vkr - 2vr + kp - p - vk + v$$
$$= 2r(k-1)(v-1) + (2r+p-v)(k-1).$$

Since each A_{iv} is an integer, we have

$$\min_{1\le i\le \infty} A_{iv} = 2r(k-1) + \mathrm{Int}\left[(2r+p-v)(k-1)/(v-1)\right]$$
$$\le 2r(k-1) + 2\gamma_{22} = 2(kc+\gamma_{22}).$$

Thus if $r_{d_v} < r$, it follows from Lemma 2.1 that for some $1 \le i \le v-1$, $\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) \ge m^*$. So now suppose $r_{d_v} = r$ and observe that since $bk = vr + p$, d must have at least $(v-p)$ treatments replicated r times i.e., $r_{d_{p+1}} = r_{d_{p+2}} = \cdots = r_{d_v} = r$. By Lemma 2.1 if $\lambda_{d_{ij}} \le \gamma_{22}$, for some $i \ne j$, $p+1 \le i, j \le v$, $\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) \ge m^*$. Thus the only way d can have

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) < \frac{2k}{kc+\gamma_{22}} \quad \forall\, i \ne j \text{ is if } \lambda_{d_{ij}} \ge \gamma_{22}+1$$

$\forall\, i \ne j$, $p+1 \le i, j \le v$.

However if this happens

$$\sum_{i=1}^{p} A_{iv} = (k-1)r_{d_v} + (k-1)\sum_{i=1}^{p} r_{d_i}(k-1) + 2\sum_{i=1}^{p}\lambda_{d_{iv}}$$
$$\le pr(k-1) + bk(k-1) - (v-p)r(k-1)$$
$$+2k[c-(v-p-1)(\gamma_{22}+1)/k]$$
$$= p[2(kc+\gamma_{22})+k+1]$$
$$+2[r(k-1)-(v-1)(\gamma_{22}+1)].$$

Therefore,

$$\min_{1\le i\le v-1} A_{iv} \le \min_{1\le i\le p} A_{iv}$$
$$\le \mathrm{Int}[2(kc+\gamma_{22})+k+1 + \frac{2[r(k-1)-(v-1)(\gamma_{22}+1)]}{p}]$$
$$\le 2(kc+\gamma_{22}),$$

whenever

$$k+1+\frac{2[r(k-1)-(v-1)(\gamma_{22}+1)]}{p} < 1.$$

Thus, for some $1 \le i \le p$, $\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) \ge \frac{2k}{kc+\gamma_{22}}$.

Hence, d^* is MV-optimal. □

Now we are going back to Section 4. We shall show now that the design d^* constructed in Section 4 is MV-optimal. The concurrence matrix $N_{d^*}N'_{d^8}$ of d^* is given by

$$\begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

and the C-matrix is given by

$$\begin{bmatrix} 2.00 & -0.667 & -0.667 & -0.667 \\ -0.667 & 1.333 & -0.333 & -0.333 \\ -0.667 & -0.333 & 1.333 & -0.333 \\ -0.667 & -0.333 & -0.333 & 1.333 \end{bmatrix}.$$

The parameters of d^* are $v=4, b=3, k=3$, $r = \mathrm{Int}\left(\frac{bk}{v}\right) = 2$, $p=1, \gamma_{11}=3$, $\gamma_{22}=1$, $\lambda_{12}=2$, $c = \frac{r(k-1)}{k} = 1.333$, $c_1 = 2$, and $c_2 = 1.333$. Clearly, d^* satisfies all the conditions of Theorem 5.1 and hence MV-optimal.

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