

Quadruple Simultaneous Fourier series Equations Involving Heat Polynomials

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Abstract: *Quadruple series equations are useful in finding the solution of four part boundary value problems of electrostatics, elasticity and other fields of mathematical physics. In the present paper, we have considered the quadruple series equations involving heat polynomials and solved them.*

1. Introduction

Quadruple series equations are useful in finding the solution of four part boundary value problems of electrostatics, elasticity and other fields of mathematical physics. Cooke [1] devised the method for finding the solution of quadruple series equations involving Fourier-Bessel series and obtained their solution by using operator theory. Recently Dwivedi and Trivedi [2] Dwivedi and Gupta [3] Dwivedi and Singh [4] considered various types of quadruple series equations involving different polynomials. In the present paper, we have considered the quadruple series equations involving heat polynomials.

2. Quadruple Simultaneous Fourier Series Equations Involving Heat Polynomials

Quadruple series equations involving heat polynomials considered here, are the generalization of dual series equations considered by Pathak [3] and corresponding triple series equations considered. Solution is obtained by reducing the problem to simultaneous Fredholm integral equations of the second kind.

3. The Equations

Here we shall consider the two sets of quadruple series equations involving heat polynomials of the first kind and second kind respectively.

(i) Quadruple Series Equations of the First Kind

Quadruple series equations of the first kind to be studied here are given as:

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma\left(\mu + \frac{1}{2} + n + p\right)} P_{n+p, \sigma}(x, -t) = f_1(x, t), \quad 0 \leq x < a \quad (1.1)$$

$$\sum_{n=0}^{\infty} \frac{t^{-n} \ell^n A_n}{\Gamma\left(\nu + \frac{1}{2} + n + p\right)} P_{n+p, \nu}(x, -t) = f_2(x, t), \quad a < x < b \quad (1.2)$$

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma\left(\mu + \frac{1}{2} + n + p\right)} P_{n+p, \sigma}(x, -t) = f_3(x, t), \quad b < x < c \quad (1.3)$$

$$\sum_{n=0}^{\infty} \frac{t^{-n} \ell^n A_n}{\Gamma\left(\nu + \frac{1}{2} + n + p\right)} P_{n+p, \nu}(x, -t) = f_4(x, t), \quad c < x < \infty \quad (1.4)$$

(ii) Quadruple Series Equations of the Second Kind

Quadruple series equations of the second kind to be analysed, here are given as:

$$\sum_{n=0}^{\infty} \frac{t^{-n} \ell^n B_n}{\Gamma\left(\nu + \frac{1}{2} + n + p\right)} P_{n+p, \nu}(x, -t) = g_1(x, t), \quad 0 \leq x < a \quad (1.5)$$

$$\sum_{n=0}^{\infty} \frac{B_n}{\Gamma\left(\mu + \frac{1}{2} + n + p\right)} P_{n+p, \sigma}(x, -t) = g_2(x, t), \quad a < x < b \quad (1.6)$$

$$\sum_{n=0}^{\infty} \frac{t^{-n} \ell^n B_n}{\Gamma\left(\nu + \frac{1}{2} + n + p\right)} P_{n+p, \nu}(x, -t) = g_3(x, t), \quad b < x < c \quad (1.7)$$

$$\sum_{n=0}^{\infty} \frac{B_n}{\Gamma\left(\mu + \frac{1}{2} + n + p\right)} P_{n+p, \sigma}(x, -t) = g_4(x, t), \quad c < x < \infty \quad (1.8)$$

In above equations $f_i(x, t)$ and $g_i(x, t)$ ($i = 1, 2, 3, 4$) are the prescribed functions $t \geq \ell > 0$ and $P_{n, \nu}(x, -t)$ is the heat polynomials.

A_n and B_n are the unknown coefficients to be determined.

4. The Solution

(i) Equations of the First Kind

In order to solve the quadruple series equations of the first kind, we set

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma\left(\mu + \frac{1}{2} + n + p\right)} P_{n+p, \sigma}(x, -t) = \begin{cases} \phi_1(x, t) & a < x < b \\ \phi_2(x, t) & c < x < \infty \end{cases} \quad (2.1)$$

where $\phi_1(x, t)$ and $\phi_2(x, t)$ are unknown functions.

Using relation (1.1) in equations (1.2), (1.3), (2.1) and (2.2), we obtain

$$A_n = \frac{\Gamma\left(\sigma + \frac{1}{2}\right)\Gamma\left(\mu + \frac{1}{2} + n + p\right)A_n}{2^{4(n+p)}(n+p)!\Gamma\left(\sigma + \frac{1}{2} + n + p\right)} \left[\int_0^a f_1(x, t) + \int_a^b \phi_1(x, t) + \int_b^c f_3(x, t) + \int_c^{\infty} \phi_2(x, t) \right] W_{n+p, \sigma}(x, t) d\Omega(x) \quad (2.3)$$

Substituting this expression for A_n in equations (1.2) and (1.4), we get

$$\sum_{n=0}^{\infty} \frac{t^{-n} e^{-n} \Gamma\left(\sigma + \frac{1}{2}\right)\Gamma\left(\mu + \frac{1}{2} + n + p\right) P_{n+p, v}(x, -t)}{2^{4(n+p)}(n+p)!\Gamma\left(\sigma + \frac{1}{2} + n + p\right)\Gamma\left(v + \frac{1}{2} + n + p\right)} \left[\int_0^a f_1(\xi, t) + \int_a^b \phi_1(x, t) + \int_b^c f_3(\xi, t) + \int_c^{\infty} \phi_2(x, t) \right] W_{n+p, \sigma}(\xi, t) d\Omega(\xi) = \begin{cases} f_2(x, t), & a < x < b \\ f_4(x, t), & c < x < \infty \end{cases} \quad (2.4)$$

Putting the value of $d\Omega(\xi)$ from equation (1.2) and changing the order of integration and summation in above equations, we get

$$\left[\int_0^a \xi^{2\sigma} f_1(\xi, t) d\xi + \int_a^b \xi^{2\sigma} \phi_1(\xi, t) d\xi + \int_b^c \xi^{2\sigma} f_3(\xi, t) d\xi + \int_c^{\infty} \xi^{2\sigma} \phi_2(\xi, t) d\xi \right] \times \frac{1}{2^{2-\sigma}} \sum_{n=0}^{\infty} \frac{\left(\frac{\ell}{t}\right)^n \Gamma\left(\mu + \frac{1}{2} + n + p\right) P_{n+p, v}(x, -t) W_{n+p, \sigma}(\xi, t)}{2^{4(n+p)}(n+p)!\Gamma\left(\sigma + \frac{1}{2} + n + p\right)\Gamma\left(v + \frac{1}{2} + n + p\right)} = \begin{cases} f_2(x, t), & a < x < b \\ f_4(x, t), & c < x < \infty \end{cases} \quad (2.7)$$

Using the summation result (1.4) in equations (2.6) and (2.7), we have

$$\int_0^a \xi^{2\sigma} f_1(\xi, t) S(x, \xi, t) d\xi + \int_a^b \xi^{2\sigma} \phi_1(\xi, t) S(x, \xi, t) d\xi + \int_b^c \xi^{2\sigma} f_3(\xi, t) S(x, \xi, t) d\xi + \int_c^{\infty} \xi^{2\sigma} \phi_2(\xi, t) S(x, \xi, t) d\xi = \begin{cases} f_2(x, t), & a < x < b \\ f_4(x, t), & c < x < \infty \end{cases} \quad (2.8)$$

$$\Rightarrow \int_a^b \xi^{2\sigma} \phi_1(\xi, t) S(x, \xi, t) d\xi + \int_c^{\infty} \xi^{2\sigma} \phi_2(\xi, t) S(x, \xi, t) d\xi$$

$$= \begin{cases} f_1(x, t), & a < x < b \\ f_2(x, t), & c < x < \infty \end{cases} \quad (2.10)$$

where

$$F_1(x, t) = f_2(x, t) - \int_0^a \xi^{2\sigma} f_1(\xi, t) S(x, \xi, t) d\xi - \int_b^c \xi^{2\sigma} f_3(\xi, t) S(x, \xi, t) d\xi \quad (2.12)$$

$$F_2(x, t) = f_4(x, t) - \int_0^a \xi^{2\sigma} f_1(\xi, t) S(x, \xi, t) d\xi - \int_b^c \xi^{2\sigma} f_3(\xi, t) S(x, \xi, t) d\xi \quad (2.13)$$

Now using the notation given by (1.6) in equation (2.10), we get

$$\int_a^b \xi^{2\sigma} \phi_1(\xi, t) \left\{ \frac{\xi^{1-2\sigma} e^{-\xi^2/4t} a^*}{x^{2v-1} \Gamma(m) \Gamma(v-\sigma+m)} S_{\omega}(\xi, x, y) \right\} d\xi + \int_c^{\infty} \xi^{2\sigma} \phi_2(\xi, t) \left\{ \frac{\xi^{2v-1} e^{-\xi^2/4t} a^*}{x^{2v-1} \Gamma(m) \Gamma(v-\sigma+m)} S_{\omega}(\xi, x, y) \right\} d\xi$$

$$= F_1(x, t), \quad a < x < b \quad (2.14)$$

$$\frac{a^*}{\Gamma(m) \Gamma(v-\sigma+m)} \left[\int_a^x \xi e^{-\xi^2/4t} \phi_1(\xi, t) S_{\xi}(\xi, x, y) d\xi + \int_x^b \xi e^{-\xi^2/4t} \phi_1(\xi, t) S_x(\xi, x, y) d\xi + \int_c^{\infty} \xi e^{\xi^2/4t} \phi_2(\xi, t) S_x(\xi, x, y) d\xi \right] = x^{2v-1} F_1(x, t), \quad a < x < b \quad (2.15)$$

Putting the value of summation in terms of integral from (1.5) in equation (2.15), we obtain

$$\int_a^x \xi e^{-\xi^2/4t} \phi_1(\xi, t) d\xi \int_0^{\xi} \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} dy + \int_x^b \xi e^{-\xi^2/4t} \phi_1(\xi, t) d\xi \int_0^x \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} dy + \int_c^{\infty} \xi e^{-\xi^2/4t} \phi_2(\xi, t) d\xi \int_0^x \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} dy = \frac{\Gamma(m) \Gamma(v-\sigma+m)}{a^* x^{1-2v}} F_1(x, t) \quad a < x < b \quad (2.16)$$

Changing the order of integration in equation (2.16), we get

$$\int_0^a \frac{\eta(y) dy}{(x^2 - y^2)^{1-v+\sigma-m}} \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi$$

$$\begin{aligned}
 & + \int_a^x \frac{\eta(y)dy}{(x^2 - y^2)^{1-\nu+\sigma-m}} \int_y^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \\
 & + \int_0^x \frac{\eta(y)dy}{(x^2 - y^2)^{1-\nu+\sigma-m}} \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \\
 & = \frac{\Gamma(m)\Gamma(\nu - \sigma + m)}{a^* x^{1-2\nu}} F_1(x, t) \quad a < x < b \quad (2.17)
 \end{aligned}$$

Assuming

$$\overline{\phi_1(y)} = \int_y^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \quad (2.18)$$

And

$$\overline{\phi_2(y)} = \int_y^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \quad (2.19)$$

Now in view of the equation (4.18) equation (4.17) can be rewritten as

$$\begin{aligned}
 & \int_a^x \frac{\eta(y)\overline{\phi_1(y)}}{(x^2 - y^2)^{1-\nu+\sigma-m}} dy = \frac{\Gamma(m)\Gamma(\nu - \sigma + m)}{a^* x^{1-2\nu}} F_1(x, t) \\
 & - \int_0^a \frac{\eta(y)dy}{(x^2 - y^2)^{1-\nu+\sigma-m}} \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \\
 & - \int_0^x \frac{\eta(y)dy}{(x^2 - y^2)^{1-\nu+\sigma-m}} \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \quad a < x < b
 \end{aligned}$$

The solution of the above equation is given as

$$\begin{aligned}
 \eta(y)\overline{\phi_1(y)} & = \frac{\text{Sin}(1 - \nu + \sigma - m)\pi}{\pi} \frac{d}{dy} \int_a^y \frac{2xdx}{(y^2 - x^2)^{\nu-\sigma+m}} \\
 & \times \left\{ \frac{\Gamma(m)\Gamma(\nu - \sigma + m)}{a^* x^{1-2\nu}} F_1(x, t) - \int_0^a \frac{\eta(z)dz}{(x^2 - z^2)^{1-\nu+\sigma-m}} \right. \\
 & \times \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi - \int_0^x \frac{\eta(z)dz}{(x^2 - z^2)^{1-\nu+\sigma-m}} \\
 & \left. \times \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right\}, \quad a < x < b \\
 \eta(y)\overline{\phi_1(y)} & = F_3(y, t) - \frac{\text{Sin}(1 - \nu + \sigma - m)\pi}{\pi} \frac{d}{dy} \int_a^y \frac{2xdx}{(y^2 - x^2)^{\nu-\sigma+m}} \\
 & \times \left\{ \int_0^a \frac{\eta(z)dz}{(x^2 - z^2)^{1-\nu+\sigma-m}} \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right. \\
 & \left. + \int_0^x \frac{\eta(z)dz}{(x^2 - z^2)^{1-\nu+\sigma-m}} \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right\} \quad a < x < b \quad (2.20)
 \end{aligned}$$

where

$$F_1(y, t) = \frac{\text{Sin}(1 - \nu + \sigma - m)\pi}{\pi} \frac{\Gamma(m)\Gamma(\nu - \sigma + m)}{a^*} \frac{d}{dy}$$

$$\int_a^y \frac{2x^{2\nu} F_1(x, t)}{(y^2 - x^2)^{\nu-\sigma+m}} - dx \quad (2.21)$$

Changing the order of integration in equation (2.20), we get

$$\begin{aligned}
 \eta(y)\overline{\phi_1(y)} & = F_3(y, t) - \frac{\text{Sin}(1 - \nu + \sigma - m)\pi}{\pi} \left\{ \eta(z)dz \times \frac{d}{dy} \right. \\
 & \int_a^y \frac{2xdx}{(y^2 - x^2)^{\nu-\sigma+m} (x^2 - z^2)^{1-\nu+\sigma-m}} \int_0^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \\
 & + \int_0^a \eta(z)dz \cdot \frac{d}{dy} \int_a^y \frac{2xdx}{(y^2 - x^2)^{\nu-\sigma+m} (x^2 - z^2)^{1-\nu+\sigma-m}} \\
 & \times \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi + \frac{d}{dy} \int_a^y \eta(z)dz \\
 & \left. \times \int_z^y \frac{2xdx}{(y^2 - x^2)^{\nu-\sigma+m} (x^2 - z^2)^{1-\nu+\sigma-m}} \int_0^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right\}
 \end{aligned}$$

$$a < x < b \quad (2.22)$$

$$\int_z^y \frac{2xdx}{(y^2 - x^2)^{\nu-\sigma+m} (x^2 - z^2)^{1-\nu+\sigma-m}} = \frac{\pi}{\text{Sin}(1 - \nu + \sigma - m)\pi} \quad (2.23)$$

In equation (4.22), we get

$$\begin{aligned}
 \eta(y)\overline{\phi_1(y)} & = F_3(y, t) - \frac{\text{Sin}(1 - \nu + \sigma - m)\pi}{\pi} \left\{ \int_0^a \eta(z)dz \right. \\
 & \times \frac{(a^2 - z^2)^{\nu-\sigma+m}}{(y^2 - z^2)(y^2 - a^2)^{\nu-\sigma+m}} \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \\
 & + \int_0^a \eta(z)dz \frac{(a^2 - z^2)^{\nu-\sigma+m}}{(y^2 - z^2)(y^2 - a^2)^{\nu-\sigma+m}} \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \\
 & \left. + \frac{d}{dy} \int_a^y \eta(z)dz \frac{\pi}{\text{Sin}(1 - \nu + \sigma - m)\pi} \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right\}
 \end{aligned}$$

$$a < x < b \quad (2.24)$$

Equations (4.18) and (4.19) are Able-type integral equations, hence their solutions are given by

$$\xi e^{-\xi^2/4t} \phi_1(\xi, t) = -\frac{\text{Sin}(1 - m)\pi}{\pi} \frac{d}{d\xi} \int_\xi^b \frac{2y\overline{\phi_1(y)}}{(y^2 - \xi^2)^m} dy \quad (2.25)$$

And

$$\xi e^{-\xi^2/4t} \phi_2(\xi, t) = -\frac{\text{Sin}(1 - m)\pi}{\pi} \frac{d}{d\xi} \int_\xi^\infty \frac{2y\overline{\phi_2(y)}}{(y^2 - \xi^2)^m} dy \quad (2.26)$$

With the help of equations (2.25) and (2.26), we obtain

$$\int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi = \frac{\text{Sin}(1-m)\pi}{\pi(a^2 - y^2)^{-m}} \int_a^b \frac{2x \overline{\phi_1(x)}}{(x^2 - a^2)^m (x^2 - y^2)} dx \quad (2.27)$$

$$\int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi = \frac{\text{Sin}(1-m)\pi}{\pi(c^2 - y^2)^{-m}} \int_c^\infty \frac{2x \overline{\phi_2(x)}}{(x^2 - c^2)^m (x^2 - y^2)} dx \quad (2.28)$$

Substituting the results (2.27) and (2.28) in equation (2.24), we have

$$\eta(y) \overline{\phi_1(y)} = F_3(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi \text{Sin}(1-m)\pi}{\pi^2(y^2 - a^2)^{v-\sigma+m}}$$

$$\left\{ \int_0^a \frac{\eta(z)(a^2 - z^2)^{v-\sigma+2m}}{(y^2 - z^2)(x^2 - z^2)} dz \times \int_a^b \frac{2x \overline{\phi_1(x)}}{(x^2 - a^2)^m} dx + \int_0^a \frac{\eta(z)(a^2 - z^2)^{v-\sigma+2m} (c^2 - z^2)^m}{(y^2 - z^2)(x^2 - z^2)} dz \times \int_c^\infty \frac{2x \overline{\phi_2(x)}}{(x^2 - a^2)^m} dx \right\}$$

$$- \frac{d}{dy} \int_a^y \frac{\eta(z)(c^2 - z^2)^m}{(x^2 - z^2)} dz \times \frac{\text{Sin}(1-m)\pi}{\pi} \int_c^\infty \frac{2x \overline{\phi_2(x)}}{(x^2 - c^2)^m} dx,$$

$$a < x < b \quad (2.29)$$

Changing the order of integration of the equation (2.29), we get

$$\eta(y) \overline{\phi_1(y)} = F_3(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi \text{Sin}(1-m)\pi}{\pi^2(y^2 - a^2)^{v-\sigma+m}}$$

$$\left\{ \int_a^b \frac{2x \overline{\phi_1(x)}}{(x^2 - a^2)^m} dx \times \int_0^a \frac{\eta(z)(a^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(x^2 - z^2)} dz + \int_c^\infty \frac{2x \overline{\phi_2(x)}}{(x^2 - a^2)^m} dx \times \int_0^a \frac{\eta(z)(a^2 - z^2)^{v-\sigma+m} (c^2 - z^2)^m}{(y^2 - z^2)(x^2 - z^2)} dz \right\}$$

$$- \frac{\text{Sin}(1-m)\pi}{\pi} \int_c^\infty \frac{2x \overline{\phi_2(x)}}{(x^2 - c^2)^m} dx \times \frac{d}{dy} \int_a^y \frac{\eta(z)(c^2 - z^2)^m}{(x^2 - z^2)} dz,$$

$$a < x < b \quad (2.30)$$

Now, above equation can be written as

$$\eta(y) \overline{\phi_1(y)} + \int_a^b M(x, y) \overline{\phi_1(y)} dx = F_3(y, t) - \int_c^\infty N(x, y) \overline{\phi_2(x)} dx,$$

$$a < x < b \quad (2.31)$$

where

$$M(x, y) = \frac{\text{Sin}(1-v+\sigma-m)\text{Sin}(1-m)\pi}{\pi^2(y^2 - a^2)^{v-\sigma+m}} \cdot \frac{2x}{(x^2 - a^2)^m}$$

$$\times \int_0^a \frac{\eta(z)(a^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(x^2 - z^2)} dz \quad (2.32)$$

$$N(x, y) = \frac{\text{Sin}(1-v+\sigma-m)\pi \text{Sin}(1-m)\pi}{\pi^2(y^2 - a^2)^{v-\sigma+m}} \cdot \frac{2x}{(x^2 - c^2)^m}$$

$$\times \int_0^a \frac{\eta(z)(a^2 - z^2)^{v-\sigma+m} (c^2 - z^2)^m}{(y^2 - z^2)(x^2 - z^2)} dz + \frac{\text{Sin}(1-m)\pi}{\pi}$$

$$\frac{2x}{(x^2 - c^2)^m} \frac{d}{dy} \int_a^y \frac{\eta(z)(c^2 - z^2)^m}{(x^2 - z^2)} dz \quad (2.33)$$

Again starting from equation (2.11), we have

$$\int_a^b \xi^{2\sigma} \phi_1(\xi, t) S(x, \xi, t) d\xi + \int_c^\infty \xi^{2\sigma} \phi_2(\xi, t) S(x, \xi, t) d\xi = F_2(x, t),$$

$$c < x < \infty \quad (2.34)$$

Using the notation given by equation (4.6), we obtain

$$\int_a^b \xi^{2\sigma} \phi_1(\xi, t) \left\{ \frac{\xi^{1-2\sigma} e^{-\xi^2/4t} a^*}{x^{2v-1} \Gamma(m) \Gamma(v-\sigma+m)} S_w(\xi, x, y) \right\} d\xi$$

$$+ \int_c^\infty \xi^{2\sigma} \phi_2(\xi, t) \left\{ \frac{\xi^{1-2\sigma} e^{-\xi^2/4t} a^*}{x^{2v-1} \Gamma(m) \Gamma(v-\sigma+m)} S_w(\xi, x, y) \right\} d\xi$$

$$= F_2(x, t), \quad c < x < \infty \quad (2.35)$$

$$\frac{a^*}{\Gamma(m) \Gamma(v-\sigma+m)} \left[\int_a^b \xi e^{-\xi^2/4t} \phi_1(\xi, t) S_\xi(\xi, x, y) d\xi + \int_c^x \xi e^{-\xi^2/4t} \phi_2(\xi, t) S_\xi(\xi, x, y) d\xi + \int_x^\infty \xi e^{-\xi^2/4t} \phi_2(\xi, t) S_\xi(\xi, x, y) d\xi \right]$$

$$= x^{2v-1} F_2(x, t), \quad c < x < \infty \quad (2.36)$$

Putting the value of summation in terms of integral from (2.5) in above equation, we get

$$\int_a^b \xi e^{-\xi^2/4t} \phi_1(\xi, t) d\xi \int_0^\xi \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} dy$$

$$+ \int_c^x \xi e^{-\xi^2/4t} \phi_2(\xi, t) d\xi \int_0^\xi \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} dy$$

$$+ \int_x^\infty \xi e^{-\xi^2/4t} \phi_2(\xi, t) d\xi \int_0^x \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} dy$$

$$= \frac{\Gamma(m) \Gamma(v-\sigma+m)}{a^* x^{1-2v}} F_2(x, t) \quad c < x < \infty \quad (2.37)$$

Changing the order of integration in equation (2.37), we get

$$\int_0^a \frac{\eta(y) dy}{(x^2 - y^2)^{1-v+\sigma-m}} \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi$$

$$\begin{aligned}
 & + \int_a^b \frac{\eta(y)dy}{(x^2 - y^2)^{1-v+\sigma-m}} \int_y^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \\
 & + \int_0^c \frac{\eta(y)dy}{(x^2 - y^2)^{1-v+\sigma-m}} \int_y^b \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \\
 & + \int_c^x \frac{\eta(y)dy}{(x^2 - y^2)^{1-v+\sigma-m}} \int_y^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \\
 & = \frac{\Gamma(m)\Gamma(v-\sigma+m)}{a^* x^{1-2v}} F_2(x, t) \quad c < x < \infty \quad (2.38)
 \end{aligned}$$

In view of the equations (2.18) and (2.19) above equation becomes

$$\begin{aligned}
 \int_c^x \frac{\eta(y)\phi_2(y)}{(x^2 - y^2)^{1-v+\sigma-m}} dy & = \frac{\Gamma(m)\Gamma(v-\sigma+m)}{a^* x^{1-2v}} F_2(x, t) \\
 - \int_0^a \frac{\eta(y)dy}{(x^2 - y^2)^{1-v+\sigma-m}} \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \\
 - \int_a^b \frac{\eta(y)dy}{(x^2 - y^2)^{1-v+\sigma-m}} \int_y^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \\
 - \int_0^c \frac{\eta(y)dy}{(x^2 - y^2)^{1-v+\sigma-m}} \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \quad c < x < \infty \quad (2.39)
 \end{aligned}$$

The solution to the equation (2.39) is given by

$$\begin{aligned}
 \eta(y)\overline{\phi_2(y)} & = \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi} \frac{d}{dy} \int_c^y \frac{2xdx}{(y^2 - x^2)^{v-\sigma+m}} \\
 & \times \left\{ \frac{\Gamma(m)\Gamma(v-\sigma+m)}{a^* x^{1-2v}} F_2(x, t) - \int_0^a \frac{\eta(z)dz}{(x^2 - y^2)^{1-v+\sigma-m}} \right. \\
 & \times \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi - \int_a^b \frac{\eta(z)dz}{(x^2 - z^2)^{1-v+\sigma-m}} \\
 & \times \int_z^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi - \int_0^c \frac{\eta(z)dz}{(x^2 - z^2)^{1-v+\sigma-m}} \\
 & \left. \times \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right\} \quad c < x < \infty \quad (2.40)
 \end{aligned}$$

Let

$$F_4(y, t) = \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi} \frac{\Gamma(m)\Gamma(v-\sigma+m)}{a^*} \frac{d}{dy}$$

$$\int_c^y \frac{2x^{2v} F_2(x, t) dx}{(y^2 - x^2)^{v-\sigma+m}} dx \quad (2.41)$$

Using (2.41) in (2.40) and changing the order of integration, we get

$$\begin{aligned}
 \eta(y)\overline{\phi_2(y)} & = F_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi} \left\{ \int_0^a \eta(z)dz \times \frac{d}{dy} \right. \\
 & \int_c^y \frac{2xdx}{(y^2 - x^2)^{v-\sigma+m}} \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \\
 & + \int_a^b \eta(z)dz \cdot \frac{d}{dy} \int_c^y \frac{2xdx}{(y^2 - x^2)^{v-\sigma+m}} \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \\
 & \left. + \int_0^c \eta(z)dz \cdot \frac{d}{dy} \int_c^y \frac{2xdx}{(y^2 - x^2)^{v-\sigma+m}} \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right\},
 \end{aligned}$$

$c < x < \infty$ (2.42)

We know that

$$\frac{d}{dy} \int_c^y \frac{2xdx}{(y^2 - x^2)^{v-\sigma+m}} \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi = \frac{(c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(y^2 - c^2)^{v-\sigma+m}} \quad (2.43)$$

Using the result (2.43) in equation (2.42), we get

$$\begin{aligned}
 \eta(y)\overline{\phi_2(y)} & = F_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi} \left\{ \int_0^a \eta(z)dz \right. \\
 & \times \frac{(c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(y^2 - c^2)^{v-\sigma+m}} \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \\
 & + \int_a^b \eta(z)dz \times \frac{(c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(y^2 - c^2)^{v-\sigma+m}} \int_z^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \\
 & \left. + \int_0^c \eta(z)dz \frac{(c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(y^2 - c^2)^{v-\sigma+m}} \times \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right\}, \\
 & \quad c < x < \infty \quad (2.44)
 \end{aligned}$$

$$\eta(y)\overline{\phi_2(y)} = F_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi(y^2 - a^2)^{v-\sigma+m}} \left\{ \int_0^a \frac{\eta(z)(c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)} dz \right.$$

$$\begin{aligned} & \times \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} dz + \int_a^b \frac{\eta(z)(c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)} dz \\ & \times \int_z^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi + \int_0^c \frac{\eta(z)(c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)} dz \\ & \times \int_c^\infty \frac{\xi e^{-\xi^2/4t} \phi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi, \end{aligned} \quad c < x < \infty \quad (2.45)$$

Substituting the values from (2.27) and (2.28) in above equation, we obtain

$$\begin{aligned} \eta(y)\overline{\phi_2(y)} = F_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi\text{Sin}(1-m)\pi}{\pi^2(y^2 - c^2)^{v-\sigma+m}} \\ \times \left\{ \int_0^a \frac{\eta(z)(a^2 - z^2)^m (c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(x^2 - z^2)} dz \int_a^b \frac{2x\overline{\phi_1(x)}}{(x^2 - a^2)^m} dx \right. \\ \left. + \int_0^c \frac{\eta(z)(c^2 - z^2)^{v-\sigma+2m}}{(y^2 - z^2)(x^2 - z^2)} dz \int_c^\infty \frac{2x\overline{\phi_2(x)}}{(x^2 - c^2)^m} dx \right\} \\ - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi(y^2 - c^2)^{v-\sigma+m}} \int_a^b \frac{\eta(z)(c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)} dz \\ \times \int_z^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi, \quad c < x < \infty \quad (2.46) \end{aligned}$$

Changing the order of integration of equation (2.46), we get

$$\begin{aligned} \eta(y)\overline{\phi_2(y)} = F_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi\text{Sin}(1-m)\pi}{\pi^2(y^2 - c^2)^{v-\sigma+m}} \\ \times \left\{ \int_a^b \frac{2x\overline{\phi_1(x)}}{(x^2 - a^2)^m} dx \int_0^a \frac{\eta(z)(a^2 - z^2)^m (c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(x^2 - z^2)} dz \right. \\ \left. + \int_c^\infty \frac{2x\overline{\phi_2(x)}}{(x^2 - c^2)^m} dx \int_0^c \frac{\eta(z)(c^2 - z^2)^{v-\sigma+2m}}{(y^2 - z^2)(x^2 - z^2)} dz \right\} \\ - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi(y^2 - c^2)^{v-\sigma+m}} \int_a^b \frac{\xi e^{-\xi^2/4t} \phi_1(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \\ \times \int_z^b \frac{\eta(z)(c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)} dz, \quad c < x < \infty \quad (2.47) \end{aligned}$$

Again using the value from equation (2.27) in equation (2.47), we get

$$\begin{aligned} \eta(y)\overline{\phi_2(y)} = F_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi\text{Sin}(1-m)\pi}{\pi^2(y^2 - c^2)^{v-\sigma+m}} \\ \times \left\{ \int_a^b \frac{2x\overline{\phi_1(x)}}{(x^2 - a^2)^m} dx \int_0^a \frac{\eta(z)(a^2 - z^2)^m (c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(x^2 - z^2)} dz \right. \end{aligned}$$

$$\begin{aligned} & + \left\{ \int_a^b \frac{2x\overline{\phi_1(x)}}{(x^2 - a^2)^m} dx \int_z^b \frac{\eta(z)(a^2 - z^2)(c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(x^2 - z^2)} dz \right. \\ & \left. + \int_c^\infty \frac{2x\overline{\phi_2(x)}}{(x^2 - c^2)^m} dx \int_0^c \frac{\eta(z)(c^2 - z^2)^{v-\sigma+2m}}{(y^2 - z^2)(x^2 - z^2)} dz \right\} \quad c < x < \infty \quad (2.48) \end{aligned}$$

Equation (2.39) can now be written as

$$\eta(y)\overline{\phi_2(y)} + \int_c^\infty P(x, y)\overline{\phi_2(x)}dx = F_4(y, t) - \int_a^b Q(x, y)\overline{\phi_1(x)}dx, \quad c < x < \infty \quad (2.49)$$

Where

$$P(x, y) = \frac{\text{Sin}(1-v+\sigma-m)\pi\text{Sin}(1-m)\pi}{\pi^2(y^2 - c^2)^{v-\sigma+m}} \cdot \frac{2x}{(x^2 - c^2)^m} \int_0^c \frac{\eta(z)(c^2 - z^2)^{v-\sigma+2m}}{(y^2 - z^2)(x^2 - z^2)} dz \quad (2.50)$$

And

$$\begin{aligned} & \left\{ \int_0^a \frac{\eta(z)(a^2 - z^2)^m (c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(x^2 - z^2)} dz \right. \\ & \left. + \int_z^b \frac{\eta(z)(a^2 - z^2)^m (c^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)(x^2 - z^2)} dz \right\} \quad (2.51) \end{aligned}$$

Equations (2.31) and (2.39) are Fredholm integral equations of the second kind which determine $\overline{\phi_1(x)}$ and $\overline{\phi_2(x)}$.

Values of $\overline{\phi_1(\xi, t)}$ and $\overline{\phi_2(\xi, t)}$ can be determined with the help of equations (2.25) and (2.26) respectively.

Finally, the coefficients A_n can be computed from equation(2.3), which satisfy the quadruple series equations involving heat polynomials, of the first kind.

$$\sum_{n=0}^{\infty} \frac{B_n}{\Gamma\left(\mu + \frac{1}{2} + n + p\right)} P_{n+p, \sigma}(x, -t) = \begin{cases} \psi_1(x, t), & 0 \leq x < a \quad (2.52) \\ \psi_2(x, t), & b < x < a \quad (2.53) \end{cases}$$

where $\psi_1(x, t)$ and $\psi_2(x, t)$ are unknown functions.

Using equation (1.1) in equations (2.52), (2.53), (1.1) in equations (2.52), (2.53), (1.6) and (1.8), we obtain

$$\begin{aligned} B_n = \frac{\Gamma\left(\sigma + \frac{1}{2}\right)\Gamma\left(\mu + \frac{1}{2} + n + p\right)}{2^{4(n+p)}(n+p)!\Gamma\left(\sigma + \frac{1}{2} + n + p\right)} \left[\int_0^a \psi_1(x, t) + \int_a^b g_2(x, t) \right. \\ \left. + \int_b^c \psi_2(x, t) + \int_c^\infty g_4(x, t) \right] W_{n+p, \sigma}(x, t) d\Omega(x) \quad (2.54) \end{aligned}$$

Substituting this expression for B_n in equations (1.5) and (1.7), we get

$$\sum_{n=0}^{\infty} \frac{t^{-n} e^{n^2} \Gamma\left(\sigma + \frac{1}{2}\right)\Gamma\left(\mu + \frac{1}{2} + n + p\right) P_{n+p, \sigma}(x, -t)}{2^{4(n+p)}(n+p)!\Gamma\left(\sigma + \frac{1}{2} + n + p\right)\Gamma\left(v + \frac{1}{2} + n + p\right)}$$

$$\left[\int_0^a \psi_1(x, t) + \int_a^b g_2(x, t) + \int_b^c \psi_2(x, t) + \int_c^\infty g_4(x, t) \right]$$

$$= \begin{cases} g_1(x, t), & 0 \leq x < a & (2.55) \\ g_3(x, t), & b < x < c & (2.56) \end{cases}$$

Using the results (1.2) in above equations and changing the order of integration and summation, we get

$$\left[\int_0^a \xi^{2\sigma} \psi_1(\xi, t) d\xi + \int_a^b \xi^{2\sigma} g_2(\xi, t) d\xi + \int_b^c \xi^{2\sigma} \psi_2(\xi, t) d\xi + \int_c^\infty \xi^{2\sigma} g_4(\xi, t) d\xi \right]$$

$$\frac{1}{2^2} \sum_{n=0}^{\infty} \frac{\left(\frac{\ell}{t}\right)^n \Gamma\left(\mu + \frac{1}{2} + n + p\right) P_{n+p, v}(x, -t)}{2^{4(n+p)} (n+p)! \Gamma\left(\sigma + \frac{1}{2} + n + p\right) \Gamma\left(v + \frac{1}{2} + n + p\right)} \times W_{n+p, \sigma}(\xi, t)$$

$$= \begin{cases} g_1(x, t), & 0 \leq x < a & (2.57) \\ g_3(x, t), & b < x < c & (2.58) \end{cases}$$

Using summation result (1.4) in equations (2.57) and (2.58), we have

$$\int_0^a \xi^{2\sigma} \psi_1(\xi, t) S(x, \xi, t) d\xi + \int_a^b \xi^{2\sigma} g_2(\xi, t) S(x, \xi, t) d\xi$$

$$+ \int_b^c \xi^{2\sigma} \psi_2(\xi, t) S(x, \xi, t) d\xi + \int_c^\infty \xi^{2\sigma} g_4(\xi, t) S(x, \xi, t) d\xi$$

$$= \begin{cases} g_1(x, t), & 0 \leq x < a & (2.59) \\ g_3(x, t), & b < x < c & (2.60) \end{cases}$$

$$\int_0^a \xi^{2\sigma} \psi_1(\xi, t) S(x, \xi, t) d\xi + \int_b^c \xi^{2\sigma} \psi_2(\xi, t) S(x, \xi, t) d\xi$$

$$= \begin{cases} g_1(x, t), & 0 \leq x < a & (2.61) \\ g_3(x, t), & b < x < c & (2.62) \end{cases}$$

where

$$G_1(x, t) = g_1(x, t) - \int_a^b \xi^{2\sigma} g_2(\xi, t) S(x, \xi, t) d\xi$$

$$- \int_c^\infty \xi^{2\sigma} g_4(\xi, t) S(x, \xi, t) d\xi \quad (2.63)$$

and

$$G_2(x, t) = g_3(x, t) - \int_a^b \xi^{2\sigma} g_2(\xi, t) S(x, \xi, t) d\xi$$

$$- \int_c^\infty \xi^{2\sigma} g_4(\xi, t) S(x, \xi, t) d\xi \quad (2.64)$$

Starting from equation (2.61), using notation given by (1.6), we get

$$\int_0^a \xi^{2\sigma} \psi_1(\xi, t) \left\{ \frac{\xi^{-2\sigma+1} e^{-\xi^2/4t} a^*}{x^{-1+2v} \Gamma(m) \Gamma(v-\sigma+m)} S_\omega(\xi, x, y) \right\} d\xi$$

$$+ \int_b^c \xi^{2\sigma} \psi_2(\xi, t) \left\{ \frac{\xi^{1-2\sigma} e^{-\xi^2/4t} a^*}{x^{2v-1} \Gamma(m) \Gamma(v-\sigma+m)} S_\omega(\xi, x, y) \right\} d\xi$$

$$= G_1(x, t), \quad 0 \leq x < a \quad (2.65)$$

$$\frac{a^*}{\Gamma(m) \Gamma(v-\sigma+m)} \left[\int_0^x \xi e^{-\xi^2/4t} \psi_1(\xi, t) S_\xi(\xi, x, y) d\xi \right]$$

$$+ \int_x^a \xi e^{-\xi^2/4t} \psi_1(\xi, t) S_x(\xi, x, y) d\xi + \int_b^c \xi e^{-\xi^2/4t} \psi_2(\xi, t) S_x(\xi, x, y) d\xi \quad (2.66)$$

Now putting the value of summation in terms of integral from (1.5) in equation (2.66), we obtain

$$\int_0^x \xi e^{-\xi^2/4t} \psi_1(\xi, t) \int_0^\xi \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} d\xi dy$$

$$+ \int_x^a \xi e^{-\xi^2/4t} \psi_1(\xi, t) \int_0^x \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} d\xi dy$$

$$+ \int_b^c \xi e^{-\xi^2/4t} \psi_2(\xi, t) \int_0^x \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} d\xi dy$$

$$= \frac{\Gamma(m) \Gamma(v-\sigma+m)}{a^* x^{1-2v}} G_1(x, t), \quad 0 \leq x < a \quad (2.67)$$

Inverting the order of integration of equation (2.67), we get

$$\int_0^x \frac{\eta(y)}{(x^2 - y^2)^{1-v+\sigma-m}} dy \int_y^a \frac{\xi e^{-\xi^2/4t} \psi_1(\xi, t)}{(x^2 - y^2)^{1-m}} d\xi$$

$$+ \int_0^x \frac{\eta(y)}{(x^2 - y^2)^{1-v+\sigma-m}} dy \int_b^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(x^2 - y^2)^{1-m}} d\xi$$

$$= \frac{\Gamma(m) \Gamma(v-\sigma+m)}{a^* x^{1-2v}} G_1(x, t), \quad 0 \leq x < a \quad (2.68)$$

Assuming,

$$\overline{\psi_1(y)} = \int_y^a \frac{\xi e^{-\xi^2/4t} \psi_1(\xi, t)}{(x^2 - y^2)^{1-m}} d\xi \quad (2.69)$$

Now equation (2.68) can be written as

$$\int_0^x \frac{\eta(y)}{(x^2 - y^2)^{1-v+\sigma-m}} dy \left\{ \overline{\psi_1(y)} + \int_b^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(x^2 - y^2)^{1-m}} d\xi \right\}$$

$$= \frac{\Gamma(m) \Gamma(v-\sigma+m)}{a^* x^{1-2v}} G_1(x, t), \quad 0 \leq x < a \quad (2.70)$$

With the help of equations (1.8), we can solve the above equation as

$$\eta(y) \left\{ \overline{\psi_1(y)} + \int_b^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(x^2 - y^2)^{1-m}} d\xi \right\}$$

$$= \frac{\sin(1-v+\sigma-m)\pi}{\pi} - \frac{d}{dy} \int_0^y \frac{2x dx}{(y^2 - x^2)^{v-\sigma+m}}$$

$$= \frac{\Gamma(m) \Gamma(v-\sigma+m)}{a^* x^{1-2v}} G_1(x, t), \quad 0 \leq x < a \quad (2.71)$$

Now equation (2.71) takes the form

$$\eta(y) \overline{\psi_1(y)} = G_3(y, t) - \eta(y) \int_b^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(x^2 - y^2)^{1-m}} d\xi, \quad 0 \leq x < a \quad (2.72)$$

Where

$$G_3(y, t) = \frac{\sin(1-v+\sigma-m)\pi \Gamma(m) \Gamma(v-\sigma+m)}{\pi a^*}$$

$$\times \frac{d}{dy} \int_0^y \frac{2x^{2v} G_1(x, t)}{(y^2 - x^2)^{v-\sigma+m}} dx \quad (2.73)$$

Solving the integral equation (2.69) as

$$\xi e^{-\xi^2/4t} \psi_1(\xi, t) = -\frac{\sin(1-m)\pi}{\pi} \frac{d}{d\xi} \int_\xi^a \frac{2y \overline{\psi_1(y)}}{(y^2 - \xi^2)^m} dy \quad (2.74)$$

With the help of equation (2.74), we obtain

$$\int_0^a \frac{\xi e^{-\xi^2/4t} \psi_1(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi = \frac{\text{Sin}(1-m)\pi}{\pi(-y^2)^{-m}} \int_0^a \frac{2x \overline{\psi_1(x)}}{(x^2 - y^2)(x^2)^m} d\xi \quad (2.75)$$

Again, let

$$\overline{\psi_2(y)} = \int_y^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \quad (2.76)$$

Solving (2.76) similarly as (2.74) and (2.75) respectively, we get

$$\xi e^{-\xi^2/4t} \psi_2(\xi, t) = -\frac{\text{Sin}(1-m)\pi}{\pi} \frac{d}{d\xi} \int_\xi^c \frac{2y \overline{\psi_2(y)}}{(y^2 - \xi^2)^m} dy \quad (2.77)$$

$$\int_b^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi = \frac{\text{Sin}(1-m)\pi}{\pi(b^2 - y^2)^{-m}} \int_b^c \frac{2x \overline{\psi_2(x)} dx}{(x^2 - b^2)^m (x^2 - y^2)} d\xi \quad (2.78)$$

Now using the equation (2.79) in (2.72), we get

$$\eta(y) \overline{\psi_1(y)} = G_3(y, t) - \frac{\text{Sin}(1-m)\pi}{\pi} \frac{\eta(y)}{(b^2 - y^2)^{-m}} \int_b^c \frac{2x \overline{\psi_2(x)} dx}{(x^2 - b^2)^m (x^2 - y^2)} \quad (2.79)$$

Equation (2.80) reduces to the following form

$$\eta(y) \overline{\psi_1(y)} = G_3(y, t) - \int_b^c R(x, y) \overline{\psi_2(y)} dx, \quad 0 \leq x < a \quad (2.80)$$

where

$$R(x, y) = \frac{\text{Sin}(1-m)\pi}{\pi} \frac{\eta(y)}{(b^2 - y^2)^{-m}} \frac{2x}{(x^2 - b^2)^m (x^2 - y^2)} \quad (2.81)$$

Again starting from equation (2.62) and using the notation (1.6), we have

$$\int_0^a \xi^{2\sigma} \psi_1(\xi, t) \left\{ \frac{\xi^{-2\sigma+1} e^{-\xi^2/4t} a^*}{x^{2v-1} \Gamma(m) \Gamma(v-\sigma+m)} S_\omega(\xi, x, y) \right\} d\xi + \int_b^c \xi^{2\sigma} \psi_2(\xi, t) \left\{ \frac{\xi^{2\sigma+1} e^{-\xi^2/4t} a^*}{x^{2v-1} \Gamma(m) \Gamma(v-\sigma+m)} S_\omega(\xi, x, y) \right\} d\xi = G_2(x, t), \quad \Phi x < c \quad (2.82)$$

$$\frac{a^*}{\Gamma(m) \Gamma(v-\sigma+m)} \int_0^a \xi e^{-\xi^2/4t} \psi_1(\xi, t) S_\xi(\xi, x, y) d\xi + \int_b^x \xi e^{-\xi^2/4t} \psi_2(\xi, t) S_\xi(\xi, x, y) d\xi + \int_x^c \xi e^{-\xi^2/4t} \psi_2(\xi, t) S_x(\xi, x, y) d\xi = x^{2v-1} G_2(x, t), \quad \Phi x < c \quad (2.83)$$

Putting the value of summation in terms of integral from (1.5) in above equation, we get

$$\int_0^a \xi e^{-\xi^2/4t} \psi_1(\xi, t) d\xi \int_0^\xi \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} dy + \int_b^x \xi e^{-\xi^2/4t} \psi_2(\xi, t) d\xi \int_0^\xi \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} dy + \int_x^c \xi e^{-\xi^2/4t} \psi_2(\xi, t) d\xi \int_0^x \eta(y) (\xi^2 - y^2)^{m-1} (x^2 - y^2)^{v-\sigma+m-1} dy$$

$$= \frac{\Gamma(m) \Gamma(v-\sigma+m)}{a^* x^{1-2v}} G_2(x, t), \quad (2.84)$$

Inverting the order of integration we get

$$\int_0^a \frac{\eta(y)}{(x^2 - y^2)^{1-v+\sigma-m}} dy \int_y^a \frac{\xi e^{-\xi^2/4t} \psi_1(\xi, t)}{(x^2 - y^2)^{1-m}} d\xi + \int_0^b \frac{\eta(y)}{(x^2 - y^2)^{1-v+\sigma-m}} dy \int_b^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(x^2 - y^2)^{1-m}} d\xi + \int_b^x \frac{\eta(y)}{(x^2 - y^2)^{1-v+\sigma-m}} dy \int_y^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(x^2 - y^2)^{1-m}} d\xi = \frac{\Gamma(m) \Gamma(v-\sigma+m)}{a^* x^{1-2v}} G_2(x, t), \quad \Phi x < c \quad (2.85)$$

Using (2.69) and (2.76) the above equation becomes

$$\int_b^x \frac{\eta(y) \overline{\psi_2(y)} dy}{(x^2 - y^2)^{1-v+\sigma-m}} = \frac{\Gamma(m) \Gamma(v-\sigma+m)}{a^* x^{1-2v}} G_2(x, t) - \int_0^a \frac{\eta(y) \overline{\psi_1(y)}}{(x^2 - y^2)^{1-v+\sigma-m}} dy - \int_0^b \frac{\eta(y)}{(x^2 - y^2)^{1-v+\sigma-m}} dy \int_b^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(\xi^2 - y^2)^{1-m}} d\xi \quad b < x < c \quad (2.86)$$

The equation (2.87) can be solved as

$$\eta(y) \overline{\psi_2(y)} = \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi} \frac{d}{dy} \int_b^y \frac{2x dx}{(y^2 - x^2)^{v-\sigma+m}} \times \left\{ \frac{\Gamma(m) \Gamma(v-\sigma-m)}{a^* x^{1-2v}} G_2(x, t) - \int_0^a \frac{\eta(z) \overline{\psi_1(z)} dz}{(x^2 - z^2)^{1-v+\sigma-m}} - \int_0^b \frac{\eta(z) dz}{(x^2 - z^2)^{1-v+\sigma-m}} \int_b^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right\}, \quad b < x < c \quad (2.87)$$

$$\eta(y) \overline{\psi_2(y)} G_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi} \frac{d}{dy} \int_b^y \frac{2x dx}{(y^2 - x^2)^{v-\sigma+m}} \times \left\{ \int_0^a \frac{\eta(z) \overline{\psi_1(z)} dz}{(x^2 - z^2)^{1-v+\sigma-m}} + \int_0^b \frac{\eta(z) dz}{(x^2 - z^2)^{1-v+\sigma-m}} \int_b^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right\}, \quad b < x < c \quad (2.88)$$

Where

$$G_4(y, t) = \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi} \frac{\Gamma(m) \Gamma(v-\sigma+m)}{a^*} \times \frac{d}{dy} \int_b^y \frac{2x^{2v} G_2(x, t)}{(y^2 - x^2)^{v-\sigma+m}} dx \quad (2.89)$$

$$\eta(y) \overline{\psi_2(y)} = G_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi} \frac{d}{dy} \int_b^y \frac{2x dx}{(y^2 - x^2)^{v-\sigma+m}} \times \left[\int_0^a \frac{dz}{(x^2 - z^2)^{1-v+\sigma-m}} \left\{ G_3(z, t) - \eta(z) \int_b^c \frac{\xi e^{-\xi^2/4t} \psi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right\} \right]$$

$$+ \int_0^b \frac{\eta(z)dz}{(x^2 - z^2)^{1-v+\sigma-m}} \int_b^c \frac{\xi e^{-\xi^2/4t} \Psi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \Bigg], \quad b < x < c \quad (2.90)$$

Breaking the last term of the above equation into two parts, we get

$$\eta(y)\overline{\Psi_2(y)} = G_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi} \frac{d}{dy} \int_b^y \frac{2xdx}{(y^2 - x^2)^{v-\sigma+m}}$$

$$\times \left[\int_0^a \frac{G_3(z, t)dz}{(x^2 - z^2)^{1-v+\sigma-m}} + \int_a^b \frac{\eta(z)dz}{(x^2 - z^2)^{1-v+\sigma-m}} \times \int_b^c \frac{\xi e^{-\xi^2/4t} \Psi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right],$$

$$b < x < c \quad (2.91)$$

Changing the order of integration, the equation (2.92) becomes

$$\eta(y)\overline{\Psi_2(y)} = G_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi} \left\{ \int_0^a G_3(z, t)dz \right.$$

$$\times \frac{d}{dy} \int_b^y \frac{2xdx}{(y^2 - x^2)^{v-\sigma+m} (x^2 - z^2)^{1-v+\sigma-m}} + \int_a^b \eta(z)dz$$

$$\left. \times \frac{d}{dy} \int_b^y \frac{2xdx}{(y^2 - x^2)^{v-\sigma+m} (x^2 - z^2)^{1-v+\sigma-m}} \times \int_b^c \frac{\xi e^{-\xi^2/4t} \Psi_2(\xi, t)}{(\xi^2 - z^2)^{1-m}} d\xi \right\}$$

$$b < x < c \quad (2.92)$$

We know that

$$\frac{d}{dy} \int_b^y \frac{2xdx}{(y^2 - x^2)^{v-\sigma+m} (x^2 - z^2)^{1-v+\sigma-m}}$$

$$= \frac{(b^2 - z^2)^{v-\sigma+m}}{(y^2 - b^2)^{v-\sigma+m} (y^2 - z^2)} \quad (2.93)$$

Using the result (2.79) and (2.94) to the equation (2.93), we get

$$\eta(y)\overline{\Psi_2(y)} = G_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi} \left\{ \int_0^a G_3(z, t)dz \right.$$

$$\times \frac{(b^2 - z^2)^{v-\sigma+m}}{(y^2 - b^2)^{v-\sigma+m} (y^2 - z^2)} + \int_a^b \eta(z)dz \times \frac{(b^2 - z^2)^{v-\sigma+m}}{(y^2 - b^2)^{v-\sigma+m} (y^2 - z^2)}$$

$$\left. \frac{\text{Sin}(1-m)\pi}{\pi(b^2 - z^2)^{-m}} \times \int_b^c \frac{2x\overline{\Psi_2(x)}}{(x^2 - b^2)^m (x^2 - z^2)} d \right\} \quad b < x < c \quad (2.94)$$

$$\eta(y)\overline{\Psi_2(y)} = G_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi(y^2 - b^2)^{v-\sigma+m}} \int_0^a \frac{G_3(z, t)(b^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)} dz$$

$$- \frac{\text{Sin}(1-v+\sigma-m)\pi\text{Sin}(1-m)\pi}{\pi^2(y^2 - b^2)^{v-\sigma+m}} + \int_a^b \frac{\eta(z)(b^2 - z^2)^{v-\sigma+2m}}{(x^2 - z^2)(y^2 - z^2)} dz$$

$$\times \int_b^c \frac{2x\overline{\Psi_2(x)}}{(x^2 - b^2)^m} d, \quad x \quad b < x < c \quad (2.95)$$

Now changing the order of integration of the last term of the equation (2.96), we get

$$\eta(y)\overline{\Psi_2(y)} = G_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)\pi}{\pi(y^2 - b^2)^{v-\sigma+m}} \int_0^a \frac{G_3(z, t)(b^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)} dz$$

$$- \frac{\text{Sin}(1-v+\sigma-m)\pi\text{Sin}(1-m)\pi}{\pi^2(y^2 - b^2)^{v-\sigma+m}} \times \int_b^c \frac{2x\overline{\Psi_2(x)}}{(x^2 - b^2)^m} dx$$

(2.96) Now equation (2.97) can be rewritten as

$$\eta(y)\overline{\Psi_2(y)} + \int_b^c S(x, y)\overline{\Psi_2(x)}dx = G_4(y, t) - \frac{\text{Sin}(1-v+\sigma-m)}{\pi(y^2 - b^2)^{v-\sigma+m}}$$

$$\int_0^a \frac{G_3(z, t)(b^2 - z^2)^{v-\sigma+m}}{(y^2 - z^2)} d, \quad z \quad b < x < c \quad (2.97)$$

Where $S(x, y)$ is the symmetric kernel

$$S(x, y) = \frac{\text{Sin}(1-v+\sigma-m)\pi\text{Sin}(1-m)\pi}{\pi^2(y^2 - b^2)^{v-\sigma+m}} \cdot \frac{2x}{(x^2 - b^2)^m}$$

$$\times \int_a^b \frac{\eta(z)(b^2 - z^2)^{v-\sigma+2m}}{(x^2 - z^2)(y^2 - z^2)} dz, \quad (2.98)$$

Equations (2.98) and (2.81) are Fredholm integral equations of the second kind determine $\overline{\Psi_2(y)}$ and

$\overline{\Psi_1(y)}$. $\Psi_1(\xi, t)$ and $\Psi_2(\xi, t)$ can be then computed from equations (2.74) and (2.78) respectively. Finally, the coefficients B_n can be calculated with the help of equation (2.54) which satisfy the equations from (1.5) to (1.8).

Particular Case

If we let $C \rightarrow \infty$ in equation (1.1) to (1.8), they reduce to the corresponding triple series equation involving heat polynomials and this solution can be shown to agree with that obtained earlier for triple series equations. Similarly, we can obtain the corresponding dual series equations involving heat polynomials.

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