Some Interesting Properties of a Subclass of Meromorphic Univalent Functions Defined by Hadamard Product

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Abstract: In the present paper, we define a new subclass \( SM_u(\tau, \gamma, \lambda, \alpha) \) of meromorphic univalent with positive coefficients defined by Hadamard product in the punctured unit disk \( U^* \). We obtain some interesting properties, like, coefficient estimates, extreme points, distortion theorem, partial sums, integral representation.

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1. Introduction

Let \( M_u \) denote the class of functions of the form:

\[
f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n,
\]

which are analytic and meromorphic univalent in the punctured unit disk \( U^* = \{z \in \mathbb{C}: 0 < |z| < 1\} \).

Let \( SM_u \) be a subclass of \( M_u \) consisting of functions of the form:

\[
f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \quad (a_n \geq 0). \tag{2}
\]

For the function \( f \in SM_u \) given by (2) and \( g \in SM_u \) defined by

\[
g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n, \quad (b_n \geq 0), \tag{3}
\]

the convolution (or Hadamard product) of \( f \) and \( g \) is defined by

\[
(f \ast g)(z) = z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n. \tag{4}
\]

We shall need to state the extended linear derivative operator of Ruscheweyh type for the function belong to the class \( SM_u \).

\[
D_{\tau}^{A,1} : SM_u \rightarrow SM_u
\]

is defined by the following convolution:

\[
D_{\tau}^{A,1} f(z) = \left( \frac{1 - \lambda}{\lambda} \right)^{A+1} f(z), \quad (\lambda > -1; f \in SM_u). \tag{5}
\]

In terms of binomial coefficients, (5) can be written as

\[
D_{\tau}^{A,1} f(z) = z^{-1} + \sum_{n=1}^{\infty} \binom{A + n}{n} a_n z^n \quad (\lambda > -1; f \in SM_u). \tag{6}
\]

The linear operator \( D_{\tau}^{A,1} \) analogous to \( D_{\tau}^{A,1} \) was considered recently by Raina and Srivastava [7] on the space of analytic and \( p \)-valent functions in \( U(\ U = U^* \cup \{0\}) \).

A function \( f \in M_u \) is said to be in the class \( M_u \) of meromorphic univalent starlike function of order \( \alpha \) if:

\[
- \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad (z \in U^*, 0 \leq \alpha < 1). \tag{7}
\]

A function \( f \in M_u \) is said to be in the class \( M_u \) of meromorphic univalent convex function of order \( \alpha \) if:

\[
- \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad (z \in U^*, 0 \leq \alpha < 1). \tag{8}
\]

Definition 1: A function \( f \in SM_u \) is said to be in the class of \( SM_u(\tau, \gamma, \lambda, \alpha) \) if it satisfies the following condition:

\[
\frac{2T}{\tau} \left( \frac{D_{\tau}^{A,1}(f \ast g)(z)}{D_{\tau}^{A,1}(f \ast g)(z)} \right)^{\gamma} + \frac{1}{\tau} < 1, \tag{9}
\]

for \( 0 < \gamma < 1, 0 < \tau < 1, 0 < \alpha < 1 \).

Atshan and Kulkarni [3], Atshan and Buti [1], Atshan and Joudah [2], Dzioik et al. [4], Khairnar and More [5] and Najafzadeh and Ebadian [6] studied meromorphic univalent and Multivalent functions for different classes.

2. Coefficient Inequality

The following theorem gives a necessary and sufficient condition for a function \( f \) to be in the class \( SM_u(\tau, \gamma, \lambda, \alpha) \).

Theorem 1: Let \( f \in SM_u \). Then \( f \in SM_u(\tau, \gamma, \lambda, \alpha) \) if and only if

\[
\sum_{n=1}^{\infty} \binom{A + n}{n} a_n b_n \leq \alpha(1 + \gamma), \tag{10}
\]

where \( 0 < \gamma < 1, 0 < \tau < 1, 0 < \alpha < 1 \).

The result is sharp for the function.
\[ f(z) = z^{-1} + \sum_{n=1}^{\infty} \frac{a(1+\gamma)}{\left(\frac{n^2}{2}(n+1) + a(1+\gamma(3+2n))b_n\right)}z^{-n} \]

**Proof:** Suppose that the inequality (10) holds true and \(|z| = 1\). Then from (9), we have

\[
\left|\frac{\sum_{n=1}^{\infty} z(n+1)\left(\frac{\lambda+n}{n}\right)a_n b_n z^n}{\left(\frac{\lambda+n}{n}\right)^2(n+1) + a(1+\gamma(3+2n))b_n}\right| \leq |a(1+\gamma)|\left(\frac{1}{r} + 2az\gamma\left(\frac{\lambda+n}{n}\right)\right)
\]

Then from (9), we have

\[
\left|\frac{\sum_{n=1}^{\infty} \frac{z}{2}D_{1,1}(f \ast g)(z) + \frac{1}{2}(D_{1,1}(f \ast g)(z))'}{(D_{1,1}(f \ast g)(z)) + \frac{1}{2}}\right| \leq |a(1+\gamma)| + 2z\gamma(z \ast (f \ast g)(z))' + \frac{(\lambda+n)}{n}a_n b_n z^n
\]

Since \(\Re(z) \leq |z|\) for all \(z \in U\), we get

\[
\Re\left(\frac{\sum_{n=1}^{\infty} \frac{z}{2}D_{1,1}(f \ast g)(z) + \frac{1}{2}(D_{1,1}(f \ast g)(z))'}{(D_{1,1}(f \ast g)(z)) + \frac{1}{2}}\right) \leq 1, \quad (12)
\]

We choose the value of \(z\) on the real axis so that \(z^nD_{1,1}(f \ast g)(z)\) is real.

Let \(z \to 1^{-}\) through real values, so we can write (12) as

\[
\Re\left(\frac{\sum_{n=1}^{\infty} \frac{z}{2}D_{1,1}(f \ast g)(z) + \frac{1}{2}(D_{1,1}(f \ast g)(z))'}{(D_{1,1}(f \ast g)(z)) + \frac{1}{2}}\right) \leq |a(1+\gamma)|.
\]

Finally, sharpness follows if we take

\[
\frac{1}{2} \sum_{n=1}^{\infty} (\lambda+n) a_n b_n z^n - a(1+\gamma)z^{-1} + 3\gamma \sum_{n=1}^{\infty} (\lambda+n) a_n b_n z^n - 2\gamma z^{-1} + \sum_{n=1}^{\infty} 2\gamma n (\lambda+n) a_n b_n z^n = 0,
\]

by hypothesis.

Hence, by maximum modulus principle, \(f \in SM_u(r, \gamma, \lambda, \alpha)\) and \(b_n \geq b_1\), \(n \geq 1\), then

\[
\frac{1}{r} - \frac{a(1+\gamma)}{(1+\lambda)(r + a(1+5\gamma))b_1}r \leq |f(z)| \leq \frac{1}{r} + \frac{a(1+\gamma)}{(1+\lambda)(r + a(1+5\gamma))b_1}r, \quad (|z| = r)
\]

We choose the value of \(z\) on the real axis so that \(z^nD_{1,1}(f \ast g)(z)\) is real.

Let \(z \to 1^{-}\) through real values, so we can write (12) as

\[
\Re\left(\frac{\sum_{n=1}^{\infty} \frac{z}{2}D_{1,1}(f \ast g)(z) + \frac{1}{2}(D_{1,1}(f \ast g)(z))'}{(D_{1,1}(f \ast g)(z)) + \frac{1}{2}}\right) \leq |a(1+\gamma)|.
\]

Finally, sharpness follows if we take

\[
\frac{1}{2} \sum_{n=1}^{\infty} (\lambda+n) a_n b_n z^n - a(1+\gamma)z^{-1} + 3\gamma \sum_{n=1}^{\infty} (\lambda+n) a_n b_n z^n - 2\gamma z^{-1} + \sum_{n=1}^{\infty} 2\gamma n (\lambda+n) a_n b_n z^n = 0,
\]

by hypothesis.

Hence, by maximum modulus principle, \(f \in SM_u(r, \gamma, \lambda, \alpha)\) and \(b_n \geq b_1\), \(n \geq 1\), then

\[
\frac{1}{r} - \frac{a(1+\gamma)}{(1+\lambda)(r + a(1+5\gamma))b_1}r \leq |f(z)| \leq \frac{1}{r} + \frac{a(1+\gamma)}{(1+\lambda)(r + a(1+5\gamma))b_1}r, \quad (|z| = r)
\]

**3. Distortion Bounds**

Next, we obtain the growth and distortion bounds for the class \(SM_u(r, \gamma, \lambda, \alpha)\).

**Theorem 2:** If \(f \in SM_u(r, \gamma, \lambda, \alpha)\) and \(b_n \geq b_1\), \(n \geq 1\), then

\[
\frac{1}{r} - \frac{a(1+\gamma)}{(1+\lambda)(r + a(1+5\gamma))b_1}r \leq |f(z)| \leq \frac{1}{r} + \frac{a(1+\gamma)}{(1+\lambda)(r + a(1+5\gamma))b_1}r, \quad (|z| = r)
\]
The result is sharp for the function:
\[ f(z) = z^{-1} + \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1}z. \] (15)

**Proof:** Since
\[ f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \]
then
\[ |f(z)| = \left| \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right| \leq \frac{1}{|z|} + \sum_{n=1}^{\infty} |a_n| |z|^n = \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n, \] (16)

Since for \( n \geq 1, \)
\[ (1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1 \leq \left( \frac{\lambda + n}{n} \right) \left( \frac{\tau}{2} (n + 1) \right) + \alpha(1 + \gamma(3 + 2n))b_n, \]
By Theorem (1), we have
\[ |f(z)| \leq \alpha(1 + \gamma). \]
That is
\[ \sum_{n=1}^{\infty} a_n \leq \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1}. \]
Using the above inequality in (16), we have
\[ |f(z)| \leq \frac{1}{r} + \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} r, \]
and
\[ |f'(z)| \geq \frac{1}{r^2} - \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} r, \]
The result is sharp for the function:
\[ f(z) = \frac{1}{z} + \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} z, \]
Similarly, we have
\[ |f'(z)| \leq \frac{1}{r^2} + \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} r, \]
and
\[ |f'(z)| \leq \frac{1}{r^2} + \frac{\alpha(1 + \gamma)}{(1 + \lambda)(\tau + \alpha(1 + 5\gamma))b_1} r. \]

**4. Partial Sums**

**Theorem 3:** Let \( f \in SM_{\lambda\alpha} \) be given by (2) and the partial sums \( S_1(z) \) and \( S_k(z) \) be defined by
\[ S_1(z) = z^{-1} \text{ and} \]
\[ S_k(z) = z^{-1} + \sum_{n=1}^{k-1} a_n z^n, (k > 1). \]
Also, suppose that
\[ \sum_{n=1}^{\infty} d_n a_n \]
\[ \leq 1, \left( d_n = \frac{(\lambda + n)(\tau)(n + 1) + \alpha(1 + \gamma(3 + 2n))}{\alpha(1 + \gamma)} \right), \] (17)
Then, we have
\[ \text{Re} \left( \frac{f(z)}{S_k(z)} \right) > 1 - \frac{1}{d_k}, \] (18)
and
\[ \text{Re} \left( \frac{f(z)}{S_k(z)} \right) > \frac{d_k}{1 + d_k}, \] (19)
Each of the bounds in (18) and (19) is the best possible for \( n \in \mathbb{N}. \)

**Proof:** For the coefficients \( d_n \) given by (17), it is not difficult to verify that
\[ d_{n+1} > d_n > 1, n = 1, 2, \ldots. \]
Therefore, by using the hypothesis (17), we have
\[ \sum_{n=1}^{k-1} a_n + d_k \sum_{n=k}^{\infty} a_n \leq \sum_{n=1}^{\infty} d_n a_n \leq 1. \] (20)
By setting
\[ g_1(z) = d_k \left( \frac{f(z)}{S_k(z)} - 1 \right), \]
and applying (20), we find that
\[ \frac{g_1(z) - 1}{g_1(z) + 1} \leq 1, \] (22)
which readily yields the assertion (18), if we take
\[ f(z) = z^{-1} - \frac{z^k}{d_k}, \] (23)
then
\[ f(z) = 1 - \frac{z^k}{d_k} \to 1 - \frac{1}{d_k} \quad (z \to 1), \]
which shows that the bound (18) is the best possible for each \( n \in \mathbb{N}. \)

Similarly, if we take
\[ g_2(z) = (1 + d_k) \left( \frac{f(z)}{S_k(z)} - 1 \right), \]
and make use of (20), we obtain
\[ \frac{g_2(z) - 1}{g_2(z) + 1} \leq 1, \] (24)
which leads us to the assertion (19). The bounds in (18) and (19) is sharp with the function given by (21).

**Theorem 4:** If \( f(z) \) of the form (2) satisfy the condition (10). Then
\[ \text{Re} \left( \frac{f(z)}{S_k(z)} \right) > 1 - \frac{k + 1}{d_{k+1}}, \]
\[ \text{Re} \left( \frac{f(z)}{S_k(z)} \right) > \frac{d_{k+1}}{k + 1 + d_{k+1}}, \] where
\[ d_n \geq \frac{(\lambda + n)^{\frac{1}{2}}}{(n + 1) + a(1 + \gamma(3 + 2n))b_n} \text{ for } n = 2, 3, ... , m \]

The bounds are sharp, with the extremal function \( f(z) \) of the form (15)

**Proof:** The proof is analogous to that Theorem 3, and we omit details.

5. Integral Representation

**Theorem 5:** Let \( f \in SM_\mu(\tau, \gamma, \lambda, \alpha) \). Then

\[
\mathcal{D}_\lambda^{1,1}(f \ast g)(z) = \exp \int_0^z \frac{\varphi(t) \alpha(1 + 3\gamma) - \frac{\tau}{2}}{\alpha(1 + 3\gamma) + 2\tau \varphi(t)} \, dt_n,
\]

where \(|\varphi(\zeta)| < 1, \zeta \in U\).

**Proof:** By putting

\[
\frac{\tau}{2} \varphi(z) + \frac{\tau}{2} = \varphi(z), \quad (|\varphi(z)| < 1, \zeta \in U).
\]

So

\[
\mathcal{D}_\lambda^{1,1}(f \ast g)(z) = \frac{\varphi(z) \alpha(1 + 3\gamma) - \frac{\tau}{2}}{\alpha(1 + 3\gamma) + 2\tau \varphi(z)}
\]

after integration, we have

\[
\log \mathcal{D}_\lambda^{1,1}(f \ast g)(z) = \int_0^z \frac{\varphi(t) \alpha(1 + 3\gamma) - \frac{\tau}{2}}{\alpha(1 + 3\gamma) + 2\tau \varphi(t)} \, dt.
\]

Therefore

\[
\mathcal{D}_\lambda^{1,1}(f \ast g)(z) = \exp \int_0^z \frac{\varphi(t) \alpha(1 + 3\gamma) - \frac{\tau}{2}}{\alpha(1 + 3\gamma) + 2\tau \varphi(t)} \, dt,
\]

and this gives the required result.

6. Extreme Points

**Theorem 6:** Let \( f_0(z) = z^{-1} \) and \( f_n(z) = z^{-1} + \alpha(1 + \gamma) \frac{(\lambda + n)^{\frac{1}{2}}}{(n + 1) + a(1 + \gamma(3 + 2n))b_n} \).

Then \( f \in SM_\mu(\tau, \gamma, \lambda, \alpha) \), if and only if it can be represented in the form

\[
f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z), \quad (\mu_n \geq 0, \sum_{n=0}^{\infty} \mu_n = 1).
\]

**Proof:** Suppose that

\[
f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z), \quad (\mu_n \geq 0, \sum_{n=0}^{\infty} \mu_n = 1).
\]

Then

\[
f(z) = \mu_0 f_0(z) + \sum_{n=1}^{\infty} \mu_n f_n(z)
\]

where

\[
\mu_n = \frac{\alpha(1 + \gamma) \mu_n}{\alpha(1 + \gamma) + \frac{(\lambda + n)^{\frac{1}{2}}}{(n + 1) + a(1 + \gamma(3 + 2n))b_n}}.
\]

Therefore

\[
\sum_{n=1}^{\infty} \mu_n \frac{(\lambda + n)^{\frac{1}{2}}}{(n + 1) + a(1 + \gamma(3 + 2n))b_n} = \frac{\alpha(1 + \gamma)}{\alpha(1 + \gamma)}
\]

So by Theorem (1), \( f \in SM_\mu(\tau, \gamma, \lambda, \alpha) \).

Conversely, suppose \( f \in SM_\mu(\tau, \gamma, \lambda, \alpha) \). By (4), we have

\[
a_n \leq \frac{\alpha(1 + \gamma)}{(\lambda + n)^{\frac{1}{2}}(n + 1) + a(1 + \gamma(3 + 2n))b_n}, \quad n \geq 1.
\]

We set

\[
\alpha_n = \frac{(\lambda + n)^{\frac{1}{2}}(n + 1) + a(1 + \gamma(3 + 2n))b_n}{\alpha(1 + \gamma)}, \quad \alpha_n \geq 1,
\]

and

\[
\mu_0 = 1 - \sum_{n=1}^{\infty} \mu_n.
\]

Then, we have

\[
f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z)
\]

Hence the results follows.

References


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