On Some Certain Properties of a New Subclass of Univalent Functions Defined by Differential Subordination Property

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Abstract: In this paper, we have studied a new subclass of univalent functions defined by differential subordination property by using the linear operator \( \mathcal{L}_{\alpha, \beta, m} \). Coefficient bounds, some properties of neighborhoods, convolution properties; Integral mean inequalities for the fractional integral for this class are obtained.

Keywords: Univalent Function, Differential Subordination, \( \phi \)-neighborhood, Integral Mean, Fractional Integral

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1. Introduction

Let \( \mathcal{S} \) be the class of all functions of from the:

\[
\mathcal{S}(\mathcal{S}) = \{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (n \in \mathbb{N}) \}
\]  
(1)

which are analytic and univalent in the open unit disk \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \).

Let \( D \) denote the subclass of \( \mathcal{S} \) containing of functions of the from:

\[
f(z) = z - \sum_{n=2}^{\infty} b_n z^n, \quad (a_n \geq 0, n \in \mathbb{N}).
\]  
(2)

The Hadamard product (or convolution) of two power series

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z)
\]  
(3)

in \( D \) is defined by:

\[
(f \ast g)(z) = f(z) \ast g(z) = \sum_{n=2}^{\infty} \left( \sum_{k=2}^{n} a_k b_{n-k} \right) z^n. 
\]  
(4)

For positive real values of \( \alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_m (\beta_j \neq 0, -1, \ldots, j = 1, 2, \ldots, m) \), the generalized hypergeometric function \( F_m(x) \) is defined by:

\[
F_m(x) = \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{m} \Gamma(n+\alpha_j)}{\prod_{j=1}^{m} \Gamma(n+\beta_j) n!} z^n, 
\]  
(5)

\( \lfloor m \rfloor \) is the pochhammer symbol defined by

\[
\lfloor m \rfloor = \begin{cases} 1, & m = 0 \, \text{or} \, 1 \, \text{or} \, 2, \\ (\alpha(n+1)(\alpha+2) \ldots (\alpha+n-1)), & \alpha \in \mathbb{N}. \end{cases}
\]  
(6)

The notation \( F_m \) is quite useful for representing many well-know functions such as the exponential, the Bessel and laguerre polynomial. Let

\[
H(\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_m): \mathbb{D} \to \mathbb{D}
\]

be a linear operator defined by

\[
H(\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_m; f(z)) = z \ast F_m(\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_m; z) \ast f(z)
\]

which is given by (1), then we see form (7), (8), (9) and (11) that

\[
F_m z = \sum_{n=2}^{\infty} W_n(a_n; \alpha_j; \beta_j; \gamma) z^n.
\]  
(7)

\[
W_n(a_n; \alpha_j; \beta_j; \gamma) = \frac{(\alpha_1)_{n-1} \ldots (\alpha_j)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_m)_{n-1} (n-1)!}.
\]  
(8)

For notational simplicity, we use shorter notation \( H_m^\alpha \) for \( H(\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_m) \).

In the sequel. It follows from (7) that

\[
H_m^\alpha f(z) = f(z), \quad H_m^\alpha z f(z) = zf(z).
\]

The linear operator \( H_m^\alpha \) is called Dozik-Srivastava operator (see [5]) introduced by Dozik and Srivastava which was subsequently extended by Dzioik and Raina [4] by using the generalized hypergeometric function, recently Srivastava et. al. [12] defined the linear operator \( \mathcal{L}_{\alpha, \beta, m} \) as follows:

\[
\mathcal{L}_{\alpha, \beta, m}^1 f(z) = (1-\lambda) H_m^\alpha f(z) + \lambda (H_m^\alpha f(z))',
\]

\[
\mathcal{L}_{\alpha, \beta, m}^2 f(z) \quad \text{and} \quad \mathcal{L}_{\alpha, \beta, m}^3 f(z) \quad \text{in general},
\]

\[
H_m^\alpha f(z) = \mathcal{L}_{\alpha, \beta, m}^1 f(z)
\]

\[
F_m(z) = \mathcal{L}_{\alpha, \beta, m}^2 f(z)
\]

\[
F_m(z) = \mathcal{L}_{\alpha, \beta, m}^3 f(z)
\]

\[
(\lfloor m \rfloor) \quad \text{and} \quad \mathcal{L}_{\alpha, \beta, m}^1 f(z) = (1-\lambda) H_m^\alpha f(z) + \lambda (H_m^\alpha f(z))', \quad (l \leq m + 1; \ i, m \in \mathbb{N}_0 = N \cup \{0\}; z \in \mathbb{U})
\]

\[
F_m(z) = \mathcal{L}_{\alpha, \beta, m}^2 f(z)
\]

\[
F_m(z) = \mathcal{L}_{\alpha, \beta, m}^3 f(z)
\]

\[
(\lfloor m \rfloor) \quad \text{and} \quad \mathcal{L}_{\alpha, \beta, m}^1 f(z) = (1-\lambda) H_m^\alpha f(z) + \lambda (H_m^\alpha f(z))', \quad (l \leq m + 1; \ i, m \in \mathbb{N}_0 = N \cup \{0\}; z \in \mathbb{U})
\]

If the function \( f(z) \) is given by (1), then we see form (7), (8), (9) and (11) that
\[ L_{\lambda,\mu,m}^{\alpha_1}(z) = 1 + Az \]

Taking \( |z| = r \), for sufficiently small \( r \) with \( 0 < r < 1 \), the denominator of (20) is positive and so it is positive for all \( r \) with \( 0 < r < 1 \), since \( w(z) \) is analytic for \( |z| < 1 \). Then, the inequality (20) yields

\[
\sum_{n=2}^{\infty} W_n^{\gamma}(\alpha_1; \lambda; \mu; m)(1 - AB)a_n r^n + (B - A - 1)r.
\]

Equivalently,

\[ \sum_{n=2}^{\infty} W_n^{\gamma}(\alpha_1; \lambda; \mu; m)(1 - AB)a_n r^n \leq (1 + A(A - B - 1))r, \]

and (17) follows upon letting \( r \rightarrow 1 \).
Conversely, for $|z| = r, 0 < r < 1$, we have $r^n < r$. That is,

$$\sum_{n=2}^{\infty} W_{n}^{+c}(\alpha; \lambda; \nu; m)(1-AB) a_n r^n$$

$$\leq \sum_{n=2}^{\infty} W_{n}^{+c}(\alpha; \lambda; \nu; m)(1-AB) a_n r^n \leq [1 + A(A-B-1)] r^n.$$

From (17), we have

$$\left| (B-A-1)z + \sum_{n=2}^{\infty} W_{n}^{+c}(\alpha; \lambda; \nu; m)(1-AB) a_n z^n \right|$$

$$\leq (B-A-1)r + \sum_{n=2}^{\infty} W_{n}^{+c}(\alpha; \lambda; \nu; m)(1-AB) a_n r^n$$

$$< \sum_{n=2}^{\infty} W_{n}^{+c}(\alpha; \lambda; \nu; m)(AB - B - AB)r$$

$$\sum_{n=2}^{\infty} W_{n}^{+c}(\alpha; \lambda; \nu; m)(AB - B - AB)z^n$$

$$+ (A^2 + B - AB) r$$

$$\leq \sum_{n=2}^{\infty} W_{n}^{+c}(\alpha; \lambda; \nu; m)(AB - B - AB) z^n$$

This prove that

$$L_{\lambda, m}^{+, c} (f(z)) < \frac{1+Az}{1+Bz}, z \in U$$

and hence $f \in K(y,c, \alpha, \lambda, \mu, m, A, B)$.

**Theorem 2:** If

$$\frac{\sum_{n=2}^{\infty} n a_n}{\sum_{n=2}^{\infty} \frac{(\alpha)_1 \cdots (\alpha)_n}{(\beta_1)_1 \cdots (\beta)_n} (1+\lambda)} \leq 1 - A \left( A - B - 1 \right),$$

then $K(y,c, \alpha, \lambda, \mu, m, A, B) \subset N_\phi (e)$. 

**Proof:** It follows from (17), that if $f \in K(y,c, \alpha, \lambda, \mu, m, A, B)$, then

$$W_{2}^{+c}(\alpha; \lambda; \nu; m)(1-AB) \sum_{n=2}^{\infty} n a_n$$

$$\leq [1 + A(A-B-1)].$$

hence

$$\frac{(\alpha)_1 \cdots (\alpha)_n}{(\beta_1)_1 \cdots (\beta)_n} (1+\lambda) \sum_{n=2}^{\infty} n a_n$$

$$\leq [1 + A(A-B-1)].$$

which implies,

$$\sum_{n=2}^{\infty} n a_n \leq \frac{(\alpha)_1 \cdots (\alpha)_n}{(\beta_1)_1 \cdots (\beta)_n} (1+\lambda) \sum_{n=2}^{\infty} n a_n$$

$$= \phi. \quad (23)$$

Using (15), we get the result.

**Definition 3:** The function $g$ defined by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n$$

is said to be member of the class $K_\phi (y,c, \alpha, \lambda, \mu, m, A, B)$ if there exists a function $f \in K(y,c, \alpha, \lambda, \mu, m, A, B)$ such that

$$\frac{g(z)}{f(z)} - 1 \leq 1 - \beta,$$

$$|z| \in U, 0 \leq \beta < 1.$$

**Theorem 3:** If $f \in K(y,c, \alpha, \lambda, \mu, m, A, B)$ and

$$\beta = 1$$

$$- \frac{W_{n}^{+c}(\alpha; \lambda; \nu; m)(1-AB)}{2 \sum_{n=2}^{\infty} W_{n}^{+c}(\alpha; \lambda; \nu; m)(1-AB) - [1 + A(A-B-1)]},$$

then $N_\phi (f) \subset K_\beta (y,c, \alpha, \lambda, \mu, m, A, B)$.

**Proof:** Let $g \in N_\phi (f)$. Then we have from (14) that

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \phi,$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\phi}{2}.$$

Also since $f \in K(y,c, \alpha, \lambda, \mu, m, A, B)$, we have from (17)

$$\sum_{n=2}^{\infty} a_n \leq \frac{1 + A(A-B-1)}{W_{n}^{+c}(\alpha; \lambda; \nu; m)(1-AB)},$$

where

$$W_{n}^{+c}(\alpha; \lambda; \nu; m) = \frac{(\alpha)_1 \cdots (\alpha)_n}{(\beta_1)_1 \cdots (\beta)_n} (1+\lambda),$$

so that

$$\frac{g(z)}{f(z)} - 1 = \frac{\sum_{n=2}^{\infty} (a_n - b_n) z^n}{z - \sum_{n=2}^{\infty} a_n z^n} \leq \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} a_n} \leq \frac{\phi}{2 \sum_{n=2}^{\infty} W_{n}^{+c}(\alpha; \lambda; \nu; m)(1-AB)} = 1 - \beta.$$

Thus by Definition (3), $g \in K_\beta (y,c, \alpha, \lambda, \mu, m, A, B)$ for $\beta$ given by (25). This completes the proof.

**3. Convolution Properties**

**Theorem 4:** Let the function $f_j (j = 1, 2)$ defined by

$$f_j (z) = z - \sum_{n=2}^{\infty} a_n j z^n \quad (a_n j \geq 0, j = 1, 2),$$

be in the class $K(y,c, \alpha, \lambda, \mu, m, A, B)$.

Then $f_1 * f_2 \in K(y,c, \alpha, \lambda, \mu, m, A, \sigma)$, where

$$\sigma = \frac{W_{n}^{+c}(\alpha; \lambda; \nu; m)(A^2 - A - 1 + [1 + A(A-B-1)]^2}{W_{n}^{+c}(\alpha; \lambda; \nu; m)(A^2 - A + 1) - [1 + A(A-B-1)]^2}.$$

**Proof:** We must find the largest $\sigma$ such that

$$\sum_{n=2}^{\infty} W_{n}^{+c}(\alpha; \lambda; \nu; m)(1-AB) \sum_{n=2}^{\infty} a_n j z^n \leq 1.$$
Since \( f_j \in K(y, c, \alpha, \lambda, t, m, A, B)(j = 1, 2) \), then
\[
\sum_{n=2}^{\infty} \frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]} a_{n,j} \leq 1,
\]
\( (j = 1, 2) \). \( (27) \)

By Cauchy-Schwarz inequality, we get
\[
\sum_{n=2}^{\infty} \frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]} \sqrt{a_{n,1}a_{n,2}} \leq 1.
\]
\( (28) \)

We went only show that
\[
\frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]} \leq \frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]}.
\]
\( (29) \)

Thus it is sufficient to show that
\[
\frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]} \leq \frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]}.
\]
\( (30) \)

which implies to
\[
\frac{\sigma}{[1 + A(A - B - 1)]} \leq \frac{A^2 + 1 - [1 + A(A - B - 1)]^2}{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB) - [1 + A(A - B - 1)]^2}.
\]
\( (31) \)

**Theorem 5:** Let the function \( f_j(j = 1, 2) \) defined by (26) be in the class \( K(y, c, \alpha, \lambda, t, m, A, B) \). Then the function \( h \) defined by
\[
h(z) = z - \sum_{n=2}^{\infty} (a_{n,1} + a_{n,2}) z^n,
\]
\( (32) \)

belong to the class \( K(y, c, \alpha, \lambda, t, m, A, B) \), where
\[
W^{\gamma+c}_n(\alpha; \lambda; \nu; m)z^2A(1 - AB)^2 - 2A[1 + A(A - B - 1)]^2.
\]
and this completes the proof.

4. Integral Mean Inequalities for the Fractional Integral

**Definition 4:** The fractional integral of order \( s \) \((s > 0)\) is defined for a function by
\[
D_z^s f(z) = \frac{1}{\Gamma(s)} \int_0^z \frac{f(t)}{(z - t)^{1-s}} dt,
\]
\( (33) \)

where the function \( f \) is analytic in a simply-connected region of the complex \( z - \) plane containing, and

\[
\varepsilon \leq \frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)z^2(1 - AB)^2 - 2A[1 + A(A - B - 1)]^2}{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB) - [1 + A(A - B - 1)]^2}.
\]

**Proof:** We must find the largest \( \varepsilon \) such that
\[
\sum_{n=2}^{\infty} \frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]} (a_{n,1}^2 + a_{n,2}^2) \leq 1.
\]

Since \( f_j \in K(y, c, \alpha, \lambda, t, m, A, B)(j = 1, 2) \), we get
\[
\sum_{n=2}^{\infty} \frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]} a_{n,1}^2 \leq 1,
\]
\( (34) \)

and
\[
\sum_{n=2}^{\infty} \frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]} a_{n,2}^2 \leq 1.
\]
\( (35) \)

Combining the inequalities (34) and (35), gives
\[
\sum_{n=2}^{\infty} \frac{1}{2} \left( \frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]} \right)^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1.
\]
\( (36) \)

But \( h \in K(y, c, \alpha, \lambda, t, m, A, B) \) if and only if
\[
\sum_{n=2}^{\infty} \frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]} a_{n,1}^2 \leq 1,
\]
\( (37) \)

the inequality (37) will be satisfied if
\[
W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB) \leq \frac{1}{[1 + A(A - B - 1)]} \frac{W^{\gamma+c}_n(\alpha; \lambda; \nu; m)(1 - AB)}{[1 + A(A - B - 1)]}.
\]
\( (38) \)

In 1925, Littlewood \[7\] proved the following subordination theorem:

**Theorem 6 (Littlewood [7]):** If \( f \) and \( g \) are analytic in \( U \) with \( f < g \), then for
\[
\mu > 0 \text{ and } z = re^{i\theta} (0 < r < 1)
\]
\[
\int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |g(z)|^\mu d\theta.
\]
Theorem 7: Let \( f \in K(y, c, \alpha_1, \lambda, \mu, a, b, A, B) \) and suppose that \( f_n \) is defined by
\[
f_n = z - \frac{[1 + A(A - B - 1)]}{W^{+e}(\alpha_1; \lambda; i; m)(1 - AB)} z^n,
\]
\((n \geq 2).\) (35)

Also let
\[
\sum_{i=2}^{\infty} (i - \eta)_{\eta+1} a_i
\]
\[
\leq \frac{[1 + A(A - B - 1)]\Gamma(n + 1)\Gamma(s + \eta + 3)}{W^{+e}(\alpha_1; \lambda; i; m)(1 - AB)\Gamma(n + s + \eta + 1)\Gamma(2 - \eta)}.
\]

for \( 0 \leq \eta \leq i, s > 0, \) where \((i - \eta)_{\eta+1}\) denote the Pochhammer symbol defined by \((i - \eta)_{\eta+1} = (i - \eta)(i - \eta + 1) \ldots i.\)

If there exists an analytic function \( q \) defined by \((q(z))^{n-1}\)
\[
= \frac{W^{+e}(\alpha_1; \lambda; i; m)(1 - AB)\Gamma(n + s + \eta + 1)}{[1 + A(A - B - 1)]\Gamma(n + 1)} \sum_{i=2}^{\infty} (i - \eta)_{\eta+1} H(i) a_i z^{-i},
\]
where \( i \geq \eta \) and
\[
H(i) = \frac{\Gamma(i - \eta)}{\Gamma(i + s + \eta + 1)}\] (38)

then, for \( z = re^{i\theta} \) and \( 0 < r < 1 \)
\[
\int_{0}^{2\pi} |D_{z}^{-s-n} f(z)|^{\mu} \, d\theta,
\]
\((s > 0, i \geq 2),\)

Proof: Let
\[
f(z) = z - \sum_{i=2}^{\infty} a_i z^{-i}.
\]

For \( \eta \geq 0 \) and Definition 4, we get
\[
D_{z}^{-s-n} f(z) = \frac{\Gamma(2)\Gamma(\eta + 1)}{\Gamma(s + \eta + 2)} \left( 1 - \sum_{i=2}^{\infty} \left( \frac{\Gamma(i + 1)\Gamma(s + \eta + 2)}{\Gamma(2)\Gamma(i + s + \eta + 1)} a_i z^{-i} \right) \right).
\]

where
\[
H(i) = \frac{\Gamma(i - 1)}{\Gamma(i + s + \eta + 1)},
\]
\((s \geq 0, i \geq 2).\)

Since \( H \) is decreasing function of \( i \), we have
\[
0 < H(i) \leq H(2).
\]

Similarly, from (35) and Definition 4, we get
\[
D_{z}^{-s-n} f(z) = \frac{\Gamma(2)\Gamma(\eta + 1)}{\Gamma(s + \eta + 2)} \left( 1 - \sum_{i=2}^{\infty} \left( \frac{\Gamma(i + 1)\Gamma(s + \eta + 2)}{\Gamma(2)\Gamma(i + s + \eta + 1)} a_i z^{-i} \right) \right).
\]

By setting
\[
\int_{0}^{2\pi} |D_{z}^{-s-n} f(z)|^{\mu} \, d\theta,
\]
\((s > 0, i \geq 2),\)

which readily yields \( w(0) = 0. \) For such a function \( q, \) we obtain
\[
\[q(z)\]^{n-1} \leq \frac{W^{+e}(\alpha_1; \lambda; i; m)(1 - AB)\Gamma(n + s + \eta + 1)}{[1 + A(A - B - 1)]\Gamma(n + 1)} \sum_{i=2}^{\infty} (i - \eta)_{\eta+1} H(i) a_i z^{-i},
\]

This completes the proof of the theorem.

By taking \( \eta = 0 \) in the Theorem 7, we have the following corollary:

Corollary 1: Let \( f \in K(y, c, \alpha_1, \lambda, \mu, a, b, A, B) \) and suppose that \( f_n \) is defined by (35). Also let
\[
\sum_{i=2}^{\infty} i a_i \leq \frac{[1 + A(A - B - 1)]\Gamma(n + 1)\Gamma(s + 3)}{W^{+e}(\alpha_1; \lambda; i; m)(1 - AB)\Gamma(s + \eta + 1)\Gamma(2)},
\]
\( n \geq 2.\)
If there exists an analytic function $q$ defined by
\[
\left( q(z) \right)^{-1} = \frac{\sum_{i=2}^{\infty} iH(i)a_i z^{i-1}}{[1+A(A-1)](\gamma(n+1))},
\]
where
\[
H(i) = \frac{\Gamma(i)}{\Gamma(i+s+1)}, \quad (s > 0, i \leq 2),
\]
then, for $z = re^{\theta}$ and $0 < r < 1$
\[
\int_{0}^{2\pi} |D^{s\phi}_z f(z)|^\mu d\theta \leq \int_{0}^{2\pi} |D^{s\phi}_z f_n(z)|^\mu d\theta, \quad (s > 0, \mu > 0).
\]

References


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