A Study on the Computation of the Determinants of a 3x3 Matrix

Awol Assen[1], J. Venkateswara Rao[2]

[1]Department of Mathematics, Faculty of Natural and Computational Science, University of Bule Hora, P.O.Box No: 144, Bule Hora, Oromia, Ethiopia

[2]Professor of Mathematics, Mekelle University Main Campus, P.O.Box No: 231, Mekelle,Tigrai , Ethiopia

Abstract: ***This manuscript endow with a new method to compute the determinants of a 3x3 matrix by describing only five entries of the determinants, where it is created a new easy scheme to compute. The main advantage of this method when compared to the other known methods (The Permutation Expansion, Sarrus's Rule, Triangle's rule, expansion by the elements of whatever row or column, Chio's condensation and Dodgson's condensation) is it is very fast (like Dardan Hajrizaj new method) to compute the determinants of a 3x3 matrix and it is very easy to use.***

Keywords: Determinant of a matrix, Permutation, Condensation method, Sarrus, Cofactor.

1. Introduction

Determinants emerged gradually during the 18^{th} century through the theory of equations in the work of Leibniz, Maclaurin, Cramer, and Laplace. By the 19^{th} century, the subject had become a mathematical area of increasing significance. Gauss (who invented the name *determinant*, Cauchy, and Cayley all produced important results on the subject, and in 1841, the German Mathematician Carl Jacobi published three major **papers** which finally brought the subject into the mathematical mainstream (Ershaidat,M.N.(2007) and Rice, A. and Torrence, E. (2007)). These(Carl Jacobi results) were important in that for the first time the definition of the determinant was made in an algorithmic way and the entries in the determinant were not specified so his results applied equally well to cases were the entries were numbers or to where they were functions. These three papers by Jacobi made the idea of a determinant widely known.

Cayley gave an explicit construction of the inverse of a matrix in terms of the determinant of the matrix. He also proved that, in the case of $2x2$ matrices, that a matrix satisfies its own characteristic equation. He stated he had checked the result for $3x3$ matrices, indicating its proof, but says:-

I have not thought it necessary to undertake the labor of a formal proof of the theorem in the general case of a matrix of any degree. An axiomatic definition of a determinant was used by Weierstrass in his lectures and, after his death; it was published in 1903 in the note *On determinant theory* (Ershaidat, M.N., 2007).

The determinant of a matrix can be thought of as a function which associates a real number with every square matrix in particular for 3x3 matrices. Determinants functions have important applications to solve systems of linear equations and also used to find the inverse of an invertible matrix in particular for a 3x3 matrix. There are several methods to find the determinant of a 3x3 matrix. The main objective of this paper is to understand these several methods and formulate a new method to find the determinant of a 3x3 matrix.

2. Co-Factor Expansion and Row Reduction Method to compute Determinants of a Matrix

2.1 Introduction to Determinants

Definition: - Every square matrix **A** with numbers as elements has associated with it a single unique number called the *determinant* of **A**, which is written $\boldsymbol{detA}$ (Jeffrey, A., 2002**)**. We denote $det(A) = |A|$ and this is most often done with the actual matrix instead of the letter representing the matrix.

2.2 The Permutation Expansion

Definition:- Determinant of *n* order will be called the sum, which has *n*! different terms $\varepsilon_{j_1 j_2 ,\dots, j_n} a_{1j_1} a_{2j_2} \dots a_{nj_n}$ which will be formed of the matrix A elements, see (Lipschutz, S. and Lipson, M.(2004); Gjonbalaj, Q. and Salihu, A. (2010) and Hajrizaj, D. (2009))

$$D = \det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{vmatrix} =$$

$$\sum_{S_n} \varepsilon_{j_1, j_2, \dots, j_n} a_{1j_1} a_{2j_2} \dots \dots a_{nj_n}$$

Where

$$\varepsilon_{j_1, j_2, \dots, j_n} = \begin{cases} +1, if\ j_1, j_2, \dots, j_n\ is\ an\ even\ permutation \\ -1, if\ j_1, j_2, \dots, j_n\ is\ an\ odd\ permutation \end{cases}$$

In base of definition, determinant of the third order (for $n = 3$) can be computed in this way

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$\varepsilon_{123}a_{11}a_{22}a_{33} + \varepsilon_{132}a_{11}a_{23}a_{32} + \varepsilon_{312}a_{13}a_{21}a_{32} + \varepsilon_{321}a_{13}a_{22}a_{31} + \varepsilon_{231}a_{12}a_{23}a_{31} + \varepsilon_{213}a_{12}a_{21}a_{33} = \varepsilon_{123}a_{11}a_{22}a_{33} + \varepsilon_{132}a_{11}a_{23}a_{32} + \varepsilon_{312}a_{13}a_{21}a_{32} + \varepsilon_{321}a_{13}a_{22}a_{31} + \varepsilon_{231}a_{12}a_{23}a_{31} + \varepsilon_{213}a_{12}a_{21}a_{33}$$

$$= a_{22}a_{11}a_{33} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}$$

Example: - Let A = $\begin{bmatrix} -2 & 10 & 2 \\ 1 & 0 & 7 \\ 0 & -3 & 5 \end{bmatrix}$ then find $\det(A)$.

Solution: - $\det(A) = \begin{vmatrix} -2 & 10 & 2 \\ 1 & 0 & 7 \\ 0 & -3 & 5 \end{vmatrix}$

$= (-2)(0)(5) - 2(7)(-3) + 2(1)(-3) - 2(0)(0) + 10(7)(0) - 10(1)(5) = -98$

2.3The Co-Factor Expansion

Definition: If A is a square matrix then the **minor of** a_{ij}, denoted by M_{ij}, which is the determinant of the sub matrix that results from removing (deleting) the i^{th} row and j^{th} column of A (Lipschutz,S and Lipson, M.(2004); Leon, J, S.(2006) and Jeffrey, A. (2002)**)**.

Example 1: - Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$M_{11}: M_{11} = det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = a_{22}a_{33} - a_{32}a_{23}$

$M_{12}: M_{12} = det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} = a_{21}a_{33} - a_{23}a_{31}$

$M_{13}: M_{13} = det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = a_{21}a_{32} - a_{31}a_{22}$ and so on.

Definition: - If A is a square matrix then the **cofactor of a_{ij}**, denoted by A_{ij}, is the number $(-1)^{i+j}M_{ij}$(Lipschutz,S and Lipson, M.(2004); Leon, J, S.(2006) and Jeffrey, A. (2002)**)**.

Definition:- The recursive definition of the determinant using cofactor expansion along the i^{th} row of A:

$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + \cdots + a_{in} = \det(A) = \sum_{k=1}^{n}(-1)^{i+k}a_{ik}A_{ik}$

The recursive definition of the determinant using cofactor expansion along the j^{th} column of A:

$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + a_{4j}A_{4j} + \cdots + a_{nj}A_{nj}$.
With sum notation: $\det(A) = \sum_{k=1}^{n}(-1)^{j+k}a_{kj}A_{kj}$ (Lipschutz,S. and Lipson, M.,2004**)**.

Example: Let A= $\begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then

find $\det(A)$ along in the entire row.

Solution: - 1) In the first row,

$detA = A_{11}a_{11} + A_{12}a_{12} + A_{13}a_{13}$

$detA = (-1)^{1+1}a_{11}detM_{11} + (-1)^{2+1}a_{12}detM_{12} + (-1)^{3+1}a_{13}detM_{13}$

$= (-1)^2a_{11}\,detM_{11} + (-1)^3a_{12}\,detM_{12} + (-1)^4a_{13}\,detM_{13}$
$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = -16$

2) In the second row, $detA = A_{21}a_{21} + A_{22}\,a_{22} + A_{23}a_{23}$

$$= -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -3\begin{vmatrix} 5 & 4 \\ 4 & 6 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix} = -16$$

3) In the 3^{rd} row,

$detA = A_{31}a_{31} + A_{32}\,a_{32} + A_{33}a_{33}$

$= a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= 5\begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} + 6\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = -16$

Note: The examples above are also true with "row" replaced by "column.

3. Calculating Determinants Using Row Reduction

The idea to use row reduction on a nxn matrix is to get it down to a row-echelon form.

Note: - The determinant associated with either an upper or a lower triangular matrix **A** of any order is easily expanded, that it reduces to the product of the terms on the leading diagonal, so the expansion of the n^{th} order lower triangular determinant with elements $a_{11}(a_{22}, \ldots, a_{nn})$ on its leading diagonal

$$\det A = \begin{vmatrix} a_{22} & \cdots & \cdots & 0 \\ a_{32} & \ddots & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix} = a_{11}(a_{22}, \ldots, a_{nn}).$$

and a corresponding result is true for a upper triangular matrix**(**Jeffrey, A.(2002) and Matthews, K.R. (1998)**)**.

Example: - Let A = $\begin{bmatrix} 3 & -1 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ then find $\det A$.

Solution:- $\det A = \begin{vmatrix} 3 & -1 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{vmatrix} = (3)(4)(1) = 12$

Here are the three row operations, how the row operations used in the row reduction process

- Interchange two rows, $R_i \leftrightarrow R_j$;
- Multiply a row by a nonzero scalar. $R_i \rightarrow cR_i$;
- Add to a row a scalar multiple of another row, $R_i \rightarrow R_i + cR_j$ (Jeffrey, A., 2002**)**.

Perform row operations until the matrix is in echelon form. As we row reduce, we need to keep in mind the **properties of the determinants.**

Example:-Use row reduction to compute the determinant of the following matrix.

A = $\begin{bmatrix} -2 & 10 & 2 \\ 1 & 0 & 7 \\ 0 & -3 & 5 \end{bmatrix}$

Solution: If we interchange rows take a minus sign onto the determinant and if we multiply a row by a scalar we'll need to multiply the new determinant by the reciprocal of the scalar.

$$\det A = \begin{vmatrix} -2 & 10 & 2 \\ 1 & 0 & 7 \\ 0 & -3 & 5 \end{vmatrix} (\mathbf{R_1 \leftrightarrow R_2}) = -\begin{vmatrix} 1 & 0 & 7 \\ -2 & 10 & 2 \\ 0 & -3 & 5 \end{vmatrix} (\mathbf{R_2 + 2R_1})$$

$$= -\begin{vmatrix} 1 & 0 & 7 \\ 0 & 10 & 16 \\ 0 & -3 & 5 \end{vmatrix} \left(\frac{1}{10}R_2\right) = -(10)\begin{vmatrix} 1 & 0 & 7 \\ 0 & 1 & \frac{8}{5} \\ 0 & -3 & 5 \end{vmatrix} R_3 + 3R_2 = -(10)\begin{vmatrix} 1 & 0 & 7 \\ 0 & 1 & \frac{8}{5} \\ 0 & 0 & \frac{49}{5} \end{vmatrix}$$

Now, we've gotten the matrix down to triangular form and so at this point we can stop and just take the determinant of that and make sure to keep the scalars that are multiplying it. Here is the final computation for this problem.

$$\det A = -10(1)(1)\left(\frac{49}{5}\right) = -98$$

Note: - Now the matrix $A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & \frac{8}{5} \\ 0 & 0 & \frac{49}{5} \end{bmatrix}$ above is an example of an upper triangular matrix (all of its non-zero entries are located on or above its main diagonal). That is $\det A$ is just the product of the entries along the main diagonal of A.

4. Condensation Methods

4.1 Dodgson's Method of "Condensation"

Definition:- The Dodgson's condensation method is a method, which determinants of the order nxn expansion in determinant of the $(n-1)x(n-1)$ order, than $(n-2)x(n-2)$ order and so on (Gjonbalaj, Q. and Salihu, A. (2010) and Hajrizaj, D.(2009)). Using the Dodgson's condensation method for the determinants of the third order, we obtain

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{12}a_{23} - a_{13}a_{22} \\ a_{21}a_{32} - a_{22}a_{31} & a_{22}a_{33} - a_{23}a_{32} \end{vmatrix}$$

$$= (a_{12}a_{22} - a_{12}a_{21}).(a_{22}a_{33} - a_{23}a_{32}) - (a_{21}a_{32} - a_{22}a_{31}).(a_{12}a_{23} - a_{13}a_{22})$$
$$= a_{22}^2a_{11}a_{33} - a_{11}a_{22}a_{23}a_{32} - a_{12}a_{21}a_{22}a_{33} + a_{21}a_{32}a_{13}a_{22} + a_{22}a_{31}a_{12}a_{23} - a_{22}^2a_{31}a_{13}$$

In base of Dodgson's condensation method the final result will be divided with a_{22} term, we have

$$|A| = a_{22}a_{11}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{22}a_{31}a_{13}$$

Example: - Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then find $\det A$.

Solution: - Using the Dodgson's condensation method for the determinants of the third order, we obtain

$$\begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 5 & 4 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} 2(1) - 3(5) & 5(2) - 1(4) \\ 3(4) - 5(1) & 1(6) - 4(2) \end{vmatrix} = \begin{vmatrix} 2-15 & 10-4 \\ 12-5 & 6-8 \end{vmatrix} = -16$$

In base of Dodgson's condensation method the final result will be divided with $a_{22} = 1$ term, so we have $|A| = \frac{-16}{1} = -16$

4.2 Algorithm of Dodgson's Method of "Condensation":

The method then consisted of four steps: see (Rice, A. and Torrence, E. (2006-2007):

1) Use elementary row and column operations to remove all zeros from the interior of A.
2) Find the 2x2 determinant for every four adjacent terms to form a new $(n-1)x(n-1)$ matrix *B*.
3) Repeat this step to produce $(n-2)x(n-2)$ matrix, and then divide each term by the corresponding entry in the interior of the original matrix A, to obtain matrix *C*.
4) Continue "condensing" the matrix down, until a single number is obtained. This final number will be det A.

4.3 Chio's Condensation Method0

Chio's condensation is a method for evaluating a nxn determinant in terms of $(n-1)x(n-1)$ determinants: **see** (Gjonbalaj, Q and Salihu, A.(2010) ,Habgood, K.and Arel, I.(2011)and Hajrizaj, D.(2009))

$$|A| = \frac{1}{a_{11}^{n-2}}\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{n2} \end{vmatrix} \\ \vdots & \vdots & \ddots & \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}$$

For $n = 3$, we obtain

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{a_{11}}\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}$$

$$= \frac{1}{a_{11}}\begin{vmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{12}a_{23} - a_{13}a_{22} \\ a_{21}a_{32} - a_{22}a_{31} & a_{22}a_{33} - a_{23}a_{32} \end{vmatrix} =$$

$$\frac{1}{a_{11}}[\,(a_{11}a_{22} - a_{12}a_{21}).(a_{11}a_{33} - a_{13}a_{31}) - (a_{11}a_{32} - a_{12}a_{31}).(a_{11}a_{23} - a_{13}a_{21})]$$

$$= \frac{1}{a_{11}}[a_{11}^2 a_{22}a_{33} - a_{11}a_{22}a_{13}a_{31} - a_{12}a_{21}a_{11}a_{33} + a_{12}a_{21}a_{13}a_{31} - a_{11}^2 a_{32}a_{23} + a_{32}\,a_{13}\,a_{21} + a_{12}a_{31}a_{11}a_{23} - a_{12}a_{31}\,a_{13}\,a_{21}$$

$$= a_{22}a_{11}a_{33} - a_{22}a_{13}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23} + a_{32}a_{13}a_{21} + a_{12}a_{31}a_{23}$$

Let us verify chio's condensation method. This method is a variation of the upper-triangle procedure, except that the triangular form is achieved implicitly by evaluating 2x2 determinants. We illustrate this method by first considering the following 3x3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant det[*A*] is evaluated by reducing the first column to upper-triangle form. Therefore, factoring a_{11} from the first row gives:

Now

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} 1 & a_{12}/a_{11} & a_{13}/a_{11} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\begin{pmatrix} -a_{21}R_1+R_2 \\ -a_{31}R_1+R_3 \end{pmatrix}$$

$$= a_{11}\begin{vmatrix} 1 & a_{12}/a_{11} & a_{13}/a_{11} \\ 0 & a_{22} - a_{21}(a_{12}/a_{11}) & a_{23} - a_{21}(a_{13}/a_{11}) \\ 0 & a_{32} - a_{31}(a_{12}/a_{11}) & a_{33} - a_{31}(a_{13}/a_{11}) \end{vmatrix}$$

But, by Laplace's Method, only one 2x2 minor is needed; hence

$$|A| = a_{11}\begin{vmatrix} a_{22} - a_{21}(a_{12}/a_{11}) & a_{23} - a_{21}(a_{13}/a_{11}) \\ a_{32} - a_{31}(a_{12}/a_{11}) & a_{33} - a_{31}(a_{13}/a_{11}) \end{vmatrix}$$

Now, if one is to factor out $1/a_{11}$ from both rows, then

$$|A| = \frac{a_{11}}{a_{11}a_{11}}\begin{vmatrix} a_{22}a_{11} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} \\ a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{31}a_{13} \end{vmatrix}$$

$$\text{OR } \det A = \frac{1}{a_{11}}\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}$$

The previous equation states that an nxn determinant can be reduced to an (n-1)x(n-1) determinant whose $(n-1)^2$ elements are the determinants of 2x2 matrices. This process can be repeated until a single 2x2 determinant is obtained.

Example: - Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then find $det A$.

Solution: - Now using chio's condensation method, we have

$$|A| = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2(1) - (3)5 & 2(2) - 3(4) \\ 2(4) - 5(5) & 2(6) - 5(4) \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} (2-15) & (4-12) \\ (8-25) & (12-20) \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -13 & -8 \\ -17 & -8 \end{vmatrix} = -16$$

The number of 2x2 determinants = $(n-1)^2 = (3-1)^2 = \mathbf{4}$

5. Methods used to compute only Determinants of third order

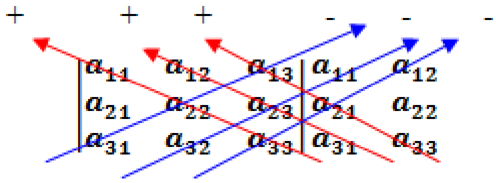
5.1 Sarrus's Rules

Sarrus's rule or **Sarrus's scheme** is a method and a memorization scheme to compute the determinant of a $3x3$ matrix. It is named after the French mathematician Pierre Frédéric Sarrus(http://en.wikipedia.org/wiki/Rule-of-Sarrus). There are four Sarrus's **rules** or **Sarrus's schemes** to compute the determinant of a $3x3$ matrix (http://common.wikimedia.org/wiki/category:sarrusrule).

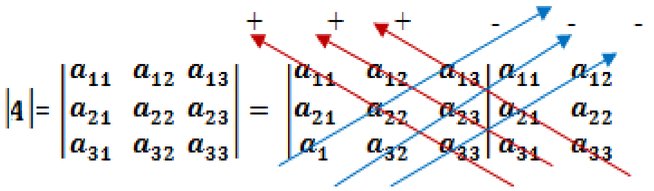
Sarrus's Scheme 1:

$If\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant can be computed by the following scheme:

(Hajrizaj, D.,2009):



Write out the first 2 columns of the matrix to the right of the 3^{rd} column, so that you have 5 columns in a row. Then add the products of the diagonals going from top to bottom and subtract the products of the diagonals going from bottom to top. These yields

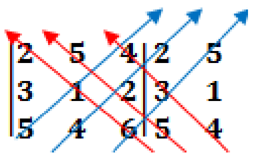


$= a_{22}a_{11}a_{33} - a_{22}a_{13}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23} + a_{32}a_{13}a_{21} + a_{12}a_{31}a_{23}$

Example: - Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then find $detA$.

Solution:

$detA = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{vmatrix} =$



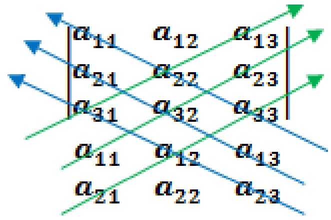
$= (2)(1)(6) + (5)(2)(5) + (4)(3)(4) - (5)(1)(4) - (4)(2)(2) - (6)(3)(5)$

$= -16$

Sarrus's Scheme 2:

$If\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant can be computed by the following scheme:

(http://common.wikimedia.org/wiki/category:sarrusrule)



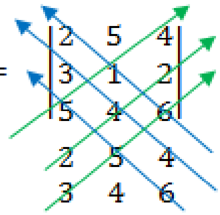
$= a_{22}a_{11}a_{33} - a_{22}a_{13}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23} + a_{32}a_{13}a_{21} + a_{12}a_{31}a_{23}$

Example: - Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then find $detA$.

Solution: In base of the Sarrus's rule 2 we have

$|A| = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{vmatrix}$

$=$

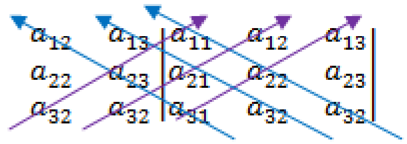


$= (2)(1)(6) + (5)(2)(5) + (4)(3)(4) - (5)(1)(4) - (4)(2)(2) - (6)(3)(5)$

$= -16$

Sarrus's Scheme 3:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. To find the determinant of a $3x3$ matrix A, write the two columns (second and third column) of A to the left of A. shows:

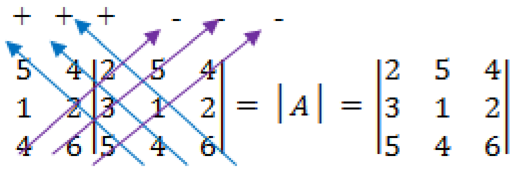


(http://common.wikimedia.org/wiki/category:sarrusrule)

$= a_{22}a_{11}a_{33} - a_{22}a_{13}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23} + a_{32}a_{13}a_{21} + c_{12}a_{31}a_{23}$

Example: - Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then find detA.

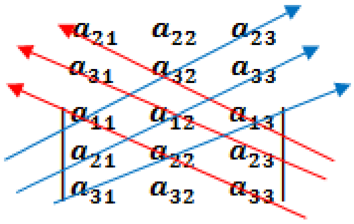
Solution: In base of the Sarrus's rule 3 we have



$= 2(1)6 + 4(3)4 + 5(2)5 - 4(2)2 - 6(3)5 - 5(1)4$
$= -16$

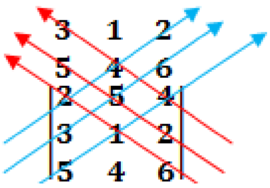
Sarrus's Scheme 4:

By similar way of above Sarrus's schemes, we have: (http://common.wikimedia.org/wiki/category:sarrusrule)



$= a_{22}a_{11}a_{33} - a_{12}a_{13}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23} + a_{32}a_{13}a_{21} + a_{12}a_{31}a_{23}$

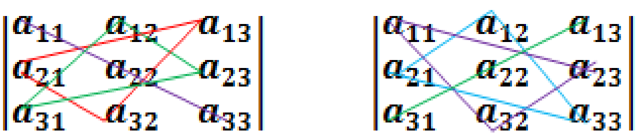
Example: -$If\ A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then $|A| = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{vmatrix} =$



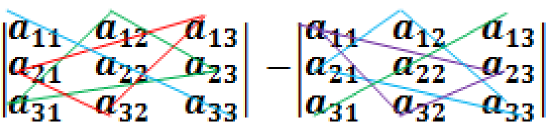
$= 2(1)6 + 4(3)4 + 5(2)5 - 4(2)2 - 6(3)5 - 5(1)4 = -16$

Triangle's Rule:

$If\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the triangle's rule will be formed as follow (Hajrizaj, D.,2009)



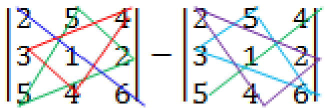
- In base of this scheme, we have:



$= a_{22}a_{11}a_{33} - a_{22}a_{13}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23} + a_{32}a_{13}a_{21} + a_{12}a_{31}a_{23}$

Example:

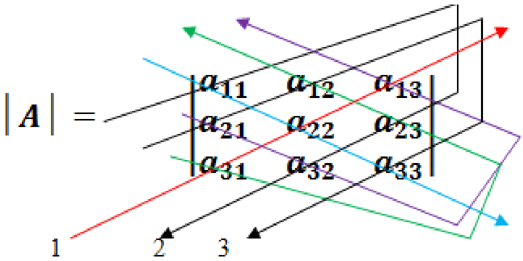
$If\ A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}, then\ |A| = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{vmatrix} =$



$= 2(1)6 + 4(3)4 + 5(2)5 - 4(2)2 - 6(3)5 - 5(1)4 = -16$

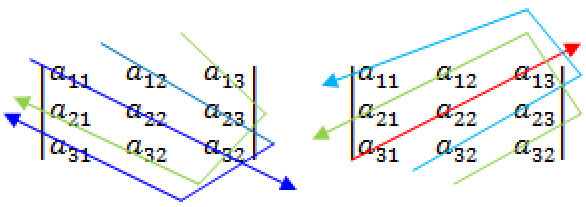
5.2 Another Scheme to compute the Determinant of a $3x3$ matrix

This scheme is represented as follow :(Hajrizaj, D.,2009)



The terms, which will be formed by the products of arrow elements of 1, 2 and 3 in above scheme become the "-"sign and the terms, which will be formed by the products of arrow elements of others arrows in above scheme become the "+"sign.

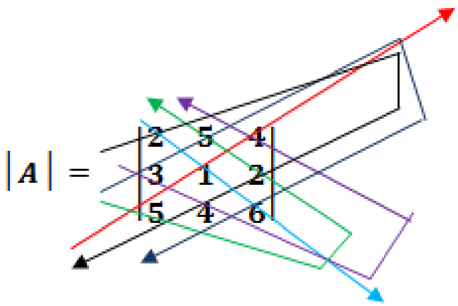
OR



The left side of this scheme is "+"sign and the right side is "-"sign (Lipschutz,S. and Lipson, M.,2004). In base of these schemes (the first scheme and the second scheme have the same result), we have

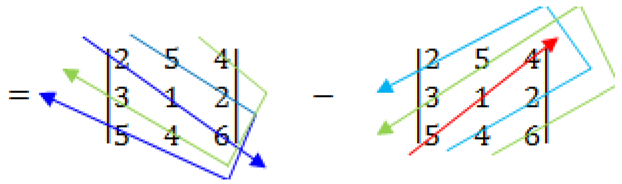
$= a_{22}a_{11}a_{33} - a_{22}a_{13}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23} + a_{32}a_{13}a_{21} + a_{12}a_{31}a_{23}$

Example: -$If\ A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then



$= 2(1)6 + 4(3)4 + 5(2)5 - 4(2)2 - 6(3)5 - 5(1) \;\; = -16$

OR



$= 2(1)6 + 4(3)4 + 5(2)5 - 4(2)2 - 6(3)5 - 5(1)4 = -16$

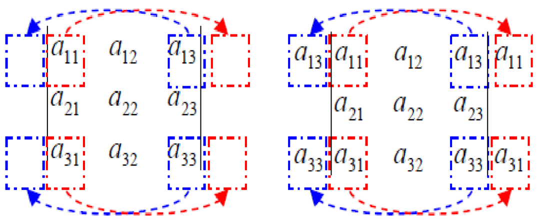
6. A new method by Dardan Hajrizaj to compute the Determinant of the third order

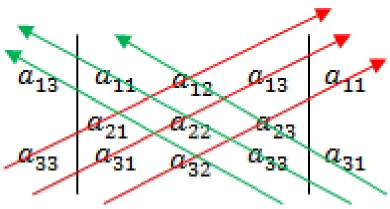
6.1 Method One: To describe this method, assume a determinant of the third order:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Let's start by describing before the first row element which lie in cutting the first row with third column and before third row element which lie in cutting the third row with third column($a_{13}$and a_{33}), as well in same manner let's describe after first and third row elements(a_{11} *and* a_{31}), respectively.

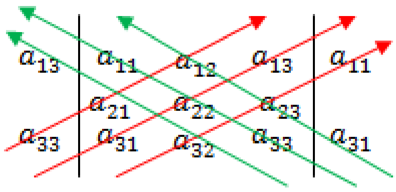
Now get such a scheme as follow: (Hajrizaj, D., 2009)





This scheme will be formed of six diagonals with three different elements of determinants. The elements products in three diagonals in left side will get the "+" sign, in the other hand the elements products in three other different diagonals in right side will get "-"sign. This will produce three terms with "+"sign and three other terms with "-"sign, which in fact presents the definition formula to compute the determinants of third order.

Proof:-While applying the new method with the above scheme, to compute the determinant of the third order,

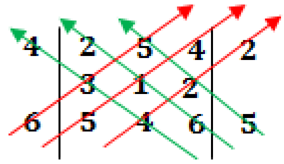


$=$

$a_{22}a_{11}a_{33} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}$

Example: - Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then find $detA$.

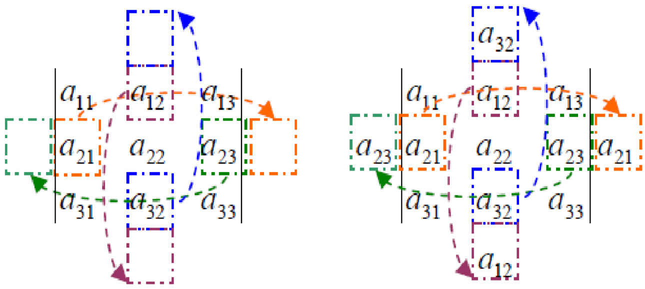
Solution:-Now using the above Scheme, to compute the determinant of A, we have

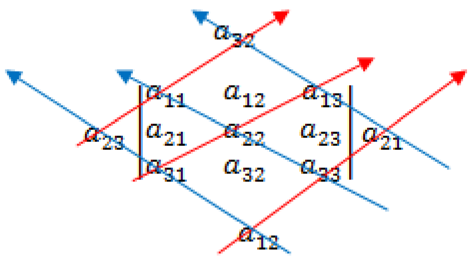


$= 4(3)(4) + (2)(1)(6) + 5(2)(5) - (6)(3)(5) - 5(1)(4) - (4)(2)(2) = -16$

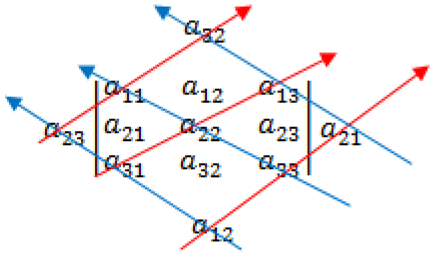
6.2 Method Two

This new method consist of scheme below, which will be formed in the same way like the preliminary scheme(above scheme) but this other scheme manipulate with elements in other rows and columns from the above scheme . (Hajrizaj, D.,2009)





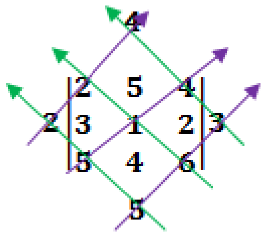
Proof:-While applying the new method with this Scheme, to compute the determinant of the third order, we have



$= a_{12}a_{23}a_{31} + a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33}$

Example: - Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then find detA.

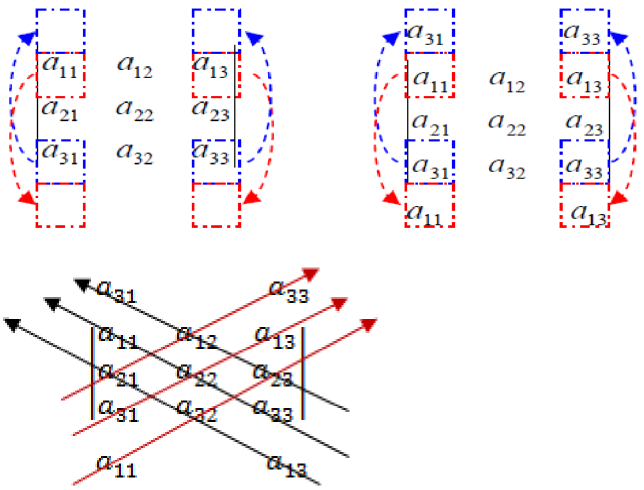
Solution: - Now using the new method above, we have



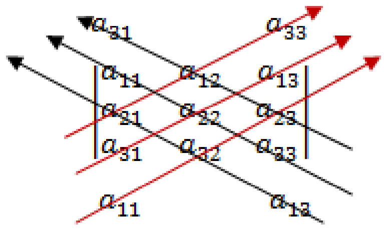
$= (-2)(2)(4) - (5)(1)(4) - (5)(6)(3) + (2)(5)(5) + (2)(1)(6) + (4)(4)(3) =$ -16

6.3 Method Three

The form of this method is shown in the following scheme: (Hajrizaj, D., 2009)



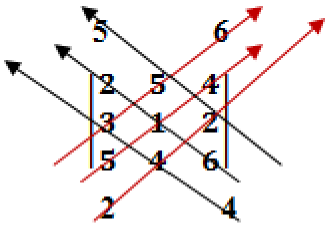
Proof: - By applying the new method with the Scheme, to calculate the determinant of the third order, we have



$= a_{12}a_{23}a_{31} + a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33}$

Example: - - Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then find detA.

Solution: By applying the new method with the Scheme, we get

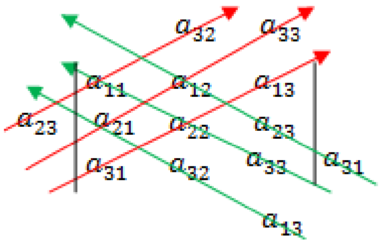


7. New methods to Compute Determinants of Order Three

Let us describe this new method to compute the determinant of the third order, assume third order determinant:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

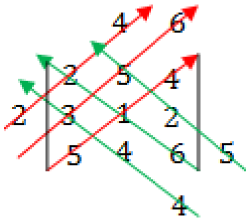
First of all, select five entries of the matrix (except the center entry because one entry must belong to two diagonal that is the center entry belong to two diagonal) and then put them in the opposite side (bottom, top, left, right side respect to our choice) of the matrix like Dardan Hajrizaj new method. Next to this, the selected position should be made a diagonal (may remain one entry from the selected five entries of the matrix) which contain three entries as shown the following scheme. Based on Dardan Hajrizaj new method, the following schemes will be formed of six diagonals with three different elements of determinants. The elements products in three diagonals in left side will get the "+" sign, in the other hand the elements products in three other different diagonals in right side will get "-"sign.



In the base of this (by similar way), we can find the following schemes and the result is the same

$$a_{22}a_{11}a_{33} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}$$

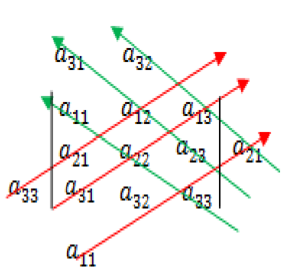
Example: - Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$ then find detA.

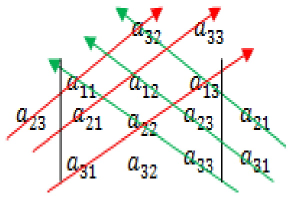


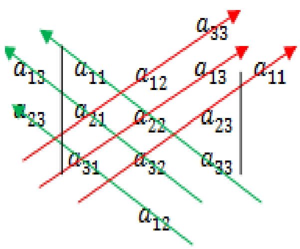
$=$
$(-2)(2)(4) - (5)(1)(4) - (5)(6)(3) + (2)(5)(5) + (2)(1)(6) + (4)(4)($
$3)$

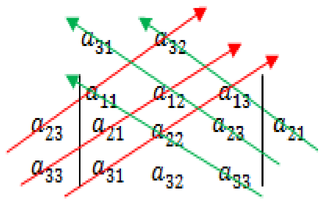
$= -16$

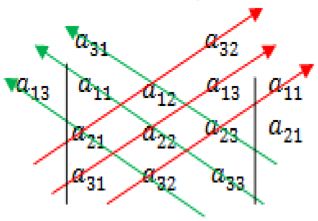
In the base of this (by similar way), we can find the following schemes and the result is the same.

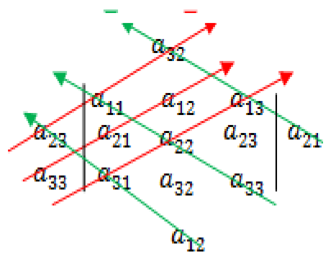


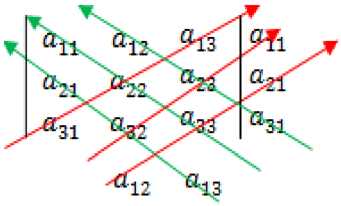


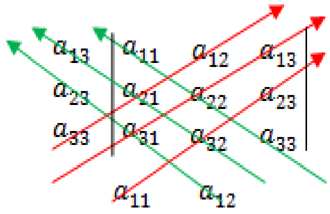


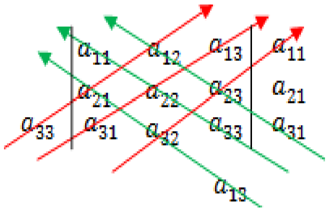


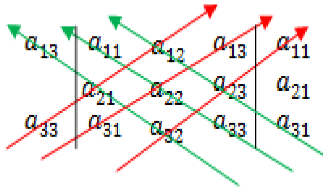












8. Conclusion

The results acquired by the other known methods which we have mentioned above are equal in determinant with different examples. Based on these results, these new methods used to find the determinants of third order and also can be used only for the third order determinant. These methods, comparing with other known methods, are the most applicable and choicest methods, based on quickness, easiness and creating different chances of computing the third order determinant. I hope, my new methods invite us for the further study in computing methods of higher than third order determinants.

9. Acknowledgement

First of all, I glorify and thank the almighty Allah for all! Next, I would like to express my sincere gratitude to my advisor, Prof. Dr. J. Venkateswara Rao for his constructive comments and advice in writing this project. Finally, I am also thankful to my wife and my friends for constant inspiration and encouragement.

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[12] http://en.wikipedia.org/wiki/Rule-of-Sarrus

Author Profile

Awol Assen: I have received my BSc. degree in 2009 from Dire Dawa University in Mathematics and M.Sc degree in Applied Mathematics from Mekelle University in 2011. I was intended International Conference on complex analysis and differential equation held in Ethiopia from June 25-29, 2012. I have served as a head of Mathematics and S tatistics department for two years and at this time I am instructor at Bule Hora University in Ethiopia.