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On Single and Multiple Currency Multifactor LIBOR Market Models: Application to Currency Options

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Abstract: ***In the mainstream Gaussian setting of log-normal forward rates, a comprehensive specification of single and multiple currency multifactor LIBOR market models is undertaken. It includes an in-depth presentation of important old and new results on correlation matrices. The evaluation of Greeks is done with the LRM algorithm. To illustrate, a Monte-Carlo simulation of European currency option prices and deltas for a cross currency multifactor LIBOR market model is tested against known analytical asymptotic expansion formulas. Concluding with a non-Gaussian copula outlook, we show the existence of single currency two-factor LIBOR market models of arbitrary dimension with arbitrary margins.***

Keywords: LIBOR market model ; forward rates ; correlation matrix ; Greeks ; currency option ; universal copula ; linear circular copula.

1. Introduction

The Black model, also called Black-76 model, first presented in [4], is a variant of the Black-Scholes option pricing model. It has been designed to price primarily options on future contracts, bond options, interest rate caps and floors, and swaptions. This model has been generalized to a class of log-normal forward models under the naming of LIBOR market (LMM) models (e.g. Fries [17], Section 17.1).

Our work is devoted to a comprehensive specification of some single and multiple currency multifactor LIBOR market models, which extend partly the models considered in Fries [17], Chapter 24, and Kawai and Jäckel [37]. Section 2 is an in-depth survey of important old and new results on correlation matrices. The presentation per se of the single and multiple currency multifactor LIBOR market models follows in the Sections 3 and 4. The evaluation of the Greeks is done with the LRM algorithm of Glasserman and Zhao [20], as proposed in Section 5. To illustrate, we test in Section 6 the Monte-Carlo simulation of European currency option prices and deltas for the cross currency multifactor LMM of Section 4.1, Case 2, against the analytical asymptotic expansion formulas by Kawai and Jäckel [37]. In the concluding Section 7, a brief outlook on the non-Gaussian setting of general LIBOR market model is given. Besides the potential use of normal variance-mean mixtures, we especially emphasize the possibility to extend LIBOR market models using copulas. In this framework, we settle in Theorem 7.1 the existence of single currency two-factor LIBOR market models of arbitrary dimension with arbitrary margins.

Before starting the specification of multifactor LIBOR market models, it is important to ask whether there is a market evidence for them. As the comment after (2.18) shows, one sometimes makes the simplifying specification of a one-factor LIBOR market model. An obvious disadvantage of single factor stochastic models is the fact that instantaneous correlation between interest rates can only be one. This contrasts with the substantial evidence that the term structure of interest rates is driven by multiple factors (three, four or even more), for which one should consult the review article by Dai and Singleton [10]. Before formulating the specification of multifactor LIBOR market models, it is important to discuss their advantages and disadvantages. Pietersz and Pelsser [50] mention five articles that compare single and multifactor models.

First, in favor of multifactor models, Longstaff, Santa-Clara and Schwartz [42] claim that short-rate models, because of miss-specified dynamics, lead to sub-optimal exercise strategies, and conclude that the costs to Wall Street firms of following single factor exercise strategies could be several billion dollars.

Second, in favor of a single factor model, Andersen and Andreasen [3] claim that the exercise strategy obtained from a properly calibrated single factor model only leads to insignificant losses when applied in a two factor model.

Third, Driessen, Klaassen and Melenberg [14] are the first to investigate hedge performance. They show that if the number of hedge instruments is equal to the number of factors, then multifactor models outperform single factor models. If, however, the large set of hedging instruments is used, which is the case in practice, then single factor models perform as well as multifactor models in terms of delta hedging of European swaptions.

Fourth, Fan, Gupta and Ritchken [16] show, for the case of the number of hedge instruments equal to the number of factors, that higher factor models perform better than lower factor models in terms of delta hedging of European swaptions and European swaption straddles. These results are thus consistent with those of Driessen et al. [14].

Fifth, Gupta and Subrahmanyam [22] show that single factor models provide accurate pricing when calibrated to the volatility smile. For hedging, however, a second stochastic factor yields better performance than a single factor model calibrated to smile.

2. An In-Depth Review of Correlation Matrices

Some important results on correlation matrices for use in the context of risk management and option pricing are reviewed and summarized. Some of them are currently widely applied. Section 2.4 presents a recent new algebraic algorithmic perspective in this area.

2.1 Valid Correlation Matrices and Spectral Decomposition

A symmetric *nxn* correlation matrix $C = (\rho_{ij})$, $\rho_{ij} = \rho_{ji} \in [-1,1]$, with unit diagonal entries $\rho_{ii} = 1$, is a *valid correlation matrix* if, and only if, it is positive semi-definite, i.e. its determinant is non-negative. This is the case provided all the eigenvalues λ_i of C are non-negative, that is $\lambda_i \geq 0, i = 1,...,n$. To ensure that a given matrix is indeed a valid correlation matrix, one often applies the *QR algorithm*, which in particular allows determination of the eigenvalues and the eigenvectors of the matrix C.

Suppose that $C = (c_1,...,c_n)$ and set $c_k^{(1)} = c_k, k = 1,...,n$. Define $u_1 = \frac{c_1}{\|c_1\|}$ and $r_{1k} = c_k \cdot u_1, k = 1,...,n$. For each $j \in \{1,...n-1\}$ define further

$$c_k^{(j+1)} = c_k^{(j)} - \left(c_k^{(j)} \cdot u_j\right) u_j, \quad k = j+1,...,n,$$

$$u_{j+1} = \frac{c_{j+1}^{(j+1)}}{\left\|c_{j+1}^{(j+1)}\right\|}, \quad r_{j+1,k} = c_k \cdot u_{j+1}, \quad k = j+1,...,n.$$

By construction the QR matrix decomposition $C = Q \cdot R$ holds, where $Q = (u_1,...,u_n)$ and $R = (r_{ij})$ is an upper triangular matrix. One distinguishes between two cases.

<u>Case 1</u>: valid correlation matrix

Through iteration of the QR algorithm, one gets the spectral decomposition

$$C = B \cdot B^T, \quad B = S \cdot \sqrt{\Lambda} \quad (2.1)$$

where $\Lambda = diag(\lambda_i)$ is the diagonal matrix of eigenvalues and S is the matrix of eigenvectors (e.g. Olver [44]). More precisely, let $C = Q \cdot R$ be the QR matrix decomposition and set $C_1 = C, Q_1 = Q, R_1 = R, C_2 = R_1 \cdot Q_1$. Through iteration set $C_k = Q_k \cdot R_k, C_{k+1} = R_k \cdot Q_k, k = 2,3,...$. The convergence properties $C_k \to \Lambda = diag(\lambda_i)$ and $Q_1 \cdot ... \cdot Q_k \to S$ as $k \to \infty$, are well-known.

<u>Remark 2.1</u>. If C is (strictly) positive definite, that is $\lambda_i > 0, i = 1,...,n$, there exists alternatively a unique lower triangular matrix L such that $C = L \cdot L^T$ (the so-called *Cholesky decomposition*). If C is only positive semi-definite, than this decomposition is not unique. Though sometimes applied in risk management and option pricing, the spectral decomposition should be preferred over the Cholesky decomposition. This is especially the case for Monte Carlo simulation in LIBOR market models.

<u>Case 2</u>: invalid correlation matrix (at least one eigenvalue $\lambda_i < 0$)

In this situation it is possible to construct a valid correlation matrix $\hat{C}$ that approximates with respect to a given norm $\|\cdot\|$ closely the given invalid correlation matrix C in the sense that the distance $\|\hat{C} - C\|$ is small enough. Among the many diverse solutions to the so-called *nearest correlation matrix problem*, the following method is especially simple and attractive.

<u>Spectral decomposition method</u> (Rebonato and Jäckel [55])

Similarly to Case 1, through iteration of the QR algorithm, one gets the diagonal matrix of eigenvalues $\Lambda = diag(\lambda_i)$ and the matrix of eigenvectors $S = (s_{ij})$ of the given invalid correlation matrix C. Consider now the diagonal matrix of non-negative elements of Λ defined by $\Lambda' = diag(\max(\lambda_i, 0))$ and the diagonal scaling matrix $T = diag(t_i)$ with non-zero elements

$$t_i = \left(\sum_{j=1}^{n} s_{ij}^2 \max(\lambda_i, 0) \right)^{-1}, \quad i = 1,...,n \quad (2.2)$$

Setting $B = \sqrt{T} \cdot S \cdot \sqrt{\Lambda'}$ one sees by construction that $\hat{C} = B \cdot B^T$ is both positive semi-definite and has unit diagonal elements, which entails a valid correlation matrix.

<u>Example 2.1</u>: spectral decomposition method for a 3x3 invalid correlation matrix

To illustrate with a numerical concrete example consider the 3x3 invalid correlation matrix

$$C = \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.9 & 1 & 0.3 \\ 0.7 & 0.3 & 1 \end{pmatrix}$$

with eigenvalue matrix

$$\Lambda = \begin{pmatrix} 2.29673 & 0 & 0 \\ 0 & 0.710625 & 0 \\ 0 & 0 & -0.00735244 \end{pmatrix}$$

and eigenvector matrix

$$S = \begin{pmatrix} 0.660 & 0.074 & 0.748 \\ 0.571 & 0.597 & -0.563 \\ 0.488 & -0.799 & -0.352 \end{pmatrix}.$$

The spectral decomposition method yields the excellent approximation

$$\hat{C} = \begin{pmatrix} 1 & 0.89402 & 0.69632 \\ 0.89402 & 1 & 0.30097 \\ 0.69632 & 0.30097 & 1 \end{pmatrix}.$$

The fewer the eigenvalues which have to be set to zero and the smaller in absolute value they are, the better the spectral decomposition method will be. The limitation of the spectral decomposition method is best illustrated as follows.

Example 2.2: limit of spectral decomposition method for a 3x3 invalid correlation matrix

Consider the 3x3 invalid correlation matrix $C = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ with eigenvalue matrix $\Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and eigenvector matrix $S = \begin{pmatrix} 0.816 & 0.000 & -0.577 \\ 0.408 & 0.707 & 0.577 \\ -0.408 & 0.707 & -0.577 \end{pmatrix}$. The spectral decomposition method yields the inaccurate approximation $\hat{C} = \begin{pmatrix} 1 & 0.5 & -0.5 \\ 0.5 & 1 & 0.5 \\ -0.5 & 0.5 & 1 \end{pmatrix}$. By the way, it is not easy to find feasible triviariate copulas with such a prescribed correlation matrix (see the concluding Section 7 for more detailed explanations).

2.2 Rank reduction of correlation matrices

The problem of rank reduction for pricing a financial product that depends on a large number of assets occurs when the assets are modeled as correlated lognormal processes. In particular, this setting includes pricing of interest rate derivatives in a LIBOR market model, and option pricing in multidimensional Black-Scholes markets without or with stochastic interest rates (e.g. author [25], [27], [28]). To illustrate, consider a model with n correlated lognormal price processes

$$dS_i / S_i = \ldots dt + \sigma_i dW_i, \quad dW_i \cdot dW_j^T = \rho_{ij} \quad (2.3)$$

under a single measure. Here S_i denotes the price of the i-th asset, σ_i its volatility and W_i denotes the associated driving Brownian motion. Brownian motions are correlated with coefficient ρ_{ij}, the correlation coefficient between the returns of assets i and j. The matrix $C = (\rho_{ij})$ should be a valid correlation matrix, that is it is positive semi-definite with unit diagonal elements. The term $\ldots dt$ stands for the drift term. To implement a scheme (2.3) by Monte Carlo simulation one applies the spectral decomposition $C = B \cdot B^T$ with B a nxn matrix. If b_i denotes the i-th row vector of B, then the decomposition reads $b_i \cdot b_j^T = \rho_{ij}$. Instead (2.3) it is more convenient to implement the scheme

$$dS_i / S_i = \ldots dt + \sigma_i \cdot (b_{i1} dZ_1 + \ldots + b_{in} dZ_n), \quad (2.4)$$

where the Z_i are now independent Brownian motions.

Now, for example, if one models a 30 year Bermudan swaption with annual call and payment dates, then the model consists of $n = 30$ annual forward LIBOR rates. For large interest rate correlation matrices, usually almost all variance (say 99%) can be attributed to only 3 to 6 Brownian motion factors. Therefore (2.4) contains a large number of almost redundant Brownian motions that cost expensive computational time to simulate. Instead of taking into account all Brownian motions, simulation is done with a smaller number of factors, say $d < n$ with d typically between 2 and 6. The scheme (2.4) is thus replaced by

$$dS_i / S_i = \ldots dt + \sigma_i \cdot (b_{i1} dZ_1 + \ldots + b_{id} dZ_d), \quad b_i \cdot b_j^T = \rho_{ij}. \quad (2.5)$$

The nxd matrix B is a decomposition of C. This approach implies that the rank of B is less than or equal to d. For financial correlation matrices this rank restriction is generally not satisfied. Therefore an approximation will be required. There are two possible ways to proceed. First, one might approximate the covariance matrix $(\sigma_i \sigma_j \rho_{ij})$. Second, one might approximate the correlation matrix (ρ_{ij}) while maintaining an exact fit to the volatilities. In practice the volatilities are usually well-known (e.g. Black-type pricing formulas or market quotes). Therefore one approximates the correlation matrix rather than the covariance matrix.

The *nearest low-rank correlation matrix problem* consists to find the nxn positive semi-definite matrix X nearest to the given nxn matrix $C = (\rho_{ij})$ through constrained optimization

$$\|C - X\| = \min. \text{ subject to } rank(X) \le d, \quad X_{ii} = 1. \quad (2.6)$$

The importance of this problem in finance has been recognized by several researchers like Rebonato [53], Higham [24] and Zhang and Wu [65]. For example, Rebonato [53] parameterizes the set of rank d correlation matrices and applies principal component analysis (PCA) to obtain a feasible point. This main simple and popular method goes back to Flury [18]. Zhang and Wu [65] have pointed out shortcomings of the PCA method and have extended it to obtain the globally nearest point. However, the latter method often does not converge. To circumvent these inconveniences, Grubisic and Pietersz [21], as well as Pietersz and Groenen [49], have considered novel techniques to solve this problem.

2.3 Full-rank and reduced rank parameterizations

We follow Brigo [5]. Given is a full-rank *nxn* valid correlation matrix $C=(\rho_{ij})$. It is characterized by $\frac{1}{2}n(n-1)$ entries, a number that can be too high for practical purposes. Therefore, a parsimonious parametric form for $C=(\rho_{ij})$ has to be found, which is based on a reduced number of parameters. There are two possibilities. One might retain a *full-rank* correlation matrix with a smaller number of parameters, or a *reduced-rank* correlation matrix.

2.3.1 Full-Rank Correlation Parameterizations

A full-rank parametric family is considered by Schoenmakers and Coffey [60]. They define correlation through a finite sequence of positive real numbers $1=c_1<c_2<\ldots<c_n$, $c_1/c_2<c_2/c_3<\ldots c_{n-1}/c_n$ and set

$$\rho_{ij}=c_i/c_j,\quad i\le j,\quad i,j=1,\ldots,n\,. \qquad (2.7)$$

This *semi-parametric correlation* structure satisfies the following conditions, which can be observed on correlation estimated from market data:

(C1) Correlation decreases for increasing maturity intervals

(C2) The forward curve tends to flatten and correlation increases with large maturities. Formally, for each integer p the correlation $\rho_{i,i+p}$ is increasing in i.

Other patterns that might be important in applications to the LIBOR market model are noted. Correlation decreases when moving away from the unit diagonal along a row or column, that is $\rho_{ij}<\rho_{i,j+1}$, $j<i$. Similarly, when moving off the diagonal in the opposite direction one has $\rho_{ij}<\rho_{i,j-1}$, $j>i$. The number of required parameters is n versus the $\frac{1}{2}n(n-1)$ entries required for a general correlation matrix. This parametric form can be characterized by a sequence of non-negative numbers $\Delta_2,\ldots,\Delta_n$ such that

$$c_i=\exp\{\sum_{j=1}^{i} j\Delta_j+\sum_{j=i+1}^{n}(i-1)\Delta_j\}\,. \qquad (2.8)$$

Parameterizations are obtained by imposing conditions on the Δ_j's. First, setting all Δ_j's equal except the last one, obtains through change of variable the following model.

<u>Stable, full rank, two-factor "increasing along sub-diagonals"</u>

$$\rho_{ij}=\exp\left\{-\frac{|i-j|}{n-1}\left(-\ln\rho_\infty+\eta\frac{n-i-j+1}{n-2}\right)\right\}, \qquad (2.9)$$
$$\rho_\infty>0,\quad 0\le\eta\le-\ln\rho_\infty$$

This parameterization is stable in the sense that relatively small movements in the c-parameters cause only relatively small changes in ρ_∞,η. Note that $\rho_\infty=\rho_{1n}$ is the correlation between forward rates with the highest distance. It is also the minimal correlation possible in the model. The parameter $\eta=\frac{1}{2}(n-1)(n-2)\Delta_{n-1}$ represents the rate of de-correlation. A derivation of formula (2.9) is found in Packham [45], Appendix B.

Second, for $n>3$ assume that $\Delta_2,\ldots,\Delta_{n-2}$ lie on a straight line and set $\Delta_{n-1}=0$.

<u>Improved, stable, full rank, two-factor parameterization</u>

$$\rho_{ij}=\exp\left\{-\frac{|i-j|}{n-1}\left(-\ln\rho_\infty+\eta\cdot f(i,j,n)\right)\right\},$$
$$\rho_\infty>0,\quad 0\le\eta\le-\ln\rho_\infty,\quad f(i,j,n) \qquad (2.10)$$
$$=\frac{i^2+j^2+ij-3ni-3nj+3i+3j+2n^2-n-4}{(n-2)(n-3)}$$

Again, one has $\rho_\infty=\rho_{1n}$. Moreover, the magnitude of concavity of the sequence $\ln c_i$ in (2.7) is decreasing to zero rather than being constant as in (2.9).

Another simple possibility is given by Rebonato [54].

<u>Classical two-factor, exponentially decreasing</u>

$$\rho_{ij}=\rho_\infty+(1-\rho_\infty)\cdot\exp\{-\beta\cdot|i-j|\}, \qquad (2.11)$$
$$-1\le\rho_\infty\le 1,\ \beta\ge 0$$

De-correlation of forward rates with increasing distance tends towards ρ_∞ rather than zero, where ρ_∞ represents asymptotically the correlation between the rates with highest distance. Since condition (C2) is not fulfilled, Rebonato [54] suggests the following modification.

<u>Modified three-factor, exponentially decreasing</u>

$$\rho_{ij}=\rho_\infty+(1-\rho_\infty)\cdot\exp\{-|i-j|\cdot(\beta-\alpha\cdot\max(i,j))\}, \qquad (2.12)$$
$$-1\le\rho_\infty\le 1,\ \beta>0,\ 0\le\alpha\le\frac{\beta}{n}$$

As pointed out by Schoenmakers and Coffey [60], this functional form may fail to define a valid correlation matrix. This problem is resolved by Rebonato [54] in two ways.

<u>Three-factor parameterization</u>

$$\rho_{ij}=\rho_\infty+(1-\rho_\infty)\cdot\exp\{-|i-j|\cdot\beta\cdot e^{-\alpha\min(i,j)}\}, \qquad (2.13)$$
$$-1\le\rho_\infty\le 1,\ \beta>0,\ \alpha\in R$$

<u>Two-factor, square root parameterization</u>

$$\rho_{ij}=\rho_\infty+(1-\rho_\infty)\cdot\exp\{-\beta\cdot\left|\sqrt{i}-\sqrt{j}\right|\}, \qquad (2.14)$$
$$-1\le\rho_\infty<1,\ \beta>0$$

2.3.2 Reduced-rank correlation parameterizations

A different parameterization approach to correlation calibration that allows for dimension reduction, as explained in Section 2.2, consists of switching to a matrix that has not full rank (e.g. Brigo [5], Section 3, and Packham [45], Section 4.4). Again, consider the spectral decomposition $C = S \cdot \Lambda \cdot S^T$ with eigenvalues $\Lambda = diag(\lambda_i)$ and orthogonal matrix of eigenvectors S. Set $A = S \cdot \sqrt{\Lambda}$ to obtain further $C = A \cdot A^T$. Replacing the nxn matrix A by a suitable d-rank nxd matrix B such that $C = B \cdot B^T$ one is led to the computationally more advantageous reduced rank simulation scheme (2.4). Of course, one is left with the problem of choosing a suitable parametric form for B such that $C = B \cdot B^T$. There exist two popular methods for this.

Principal component analysis (PCA)

Let $\Lambda^{(d)}$ be the matrix Λ with the smallest $n-d$ eigenvalues set equal to zero and define $B^{(d)} = S \cdot \sqrt{\Lambda^{(d)}}$, $\overline{C}^{(d)} = \left(\overline{\rho}_{ij}^{(d)}\right) = B^{(d)} \cdot B^{(d)T}$. To obtain a valid correlation matrix $C^{(d)} = \left(\rho_{ij}^{(d)}\right)$ rescale the entries such that

$$\rho_{ij}^{(d)} = \overline{\rho}_{ij}^{(d)} / \sqrt{\overline{\rho}_{ii}^{(d)} \overline{\rho}_{jj}^{(d)}}. \quad (2.15)$$

The eigenvector of C with highest eigenvalue is called *principal component* and has the most significant influence. Taking the d eigenvectors with highest eigenvalues results in a d-rank correlation matrix with minimum loss of information. For example, Alexander and Lvov [2] use PCA with $d = 3$ and classify the corresponding eigenvectors into trend, tilt and curvature ones. For more information about PCA the interested reader is referred to the recent overview by Abdi and Williams [1] (see also Joliffe [34]).

Hyper-sphere decomposition (Rebonato [53])

A d-rank correlation matrix is defined by $C(\theta)^{(d)} = \left(\rho(\theta)_{ij}^{(d)}\right) = B(\theta)^{(d)} \cdot B(\theta)^{(d)T}$ with

$$b(\theta)_{ij} = \begin{cases} \cos\theta_{ij} \cdot \prod_{k=1}^{j-1} \sin\theta_{ik}, & j = 1,\ldots,d-1, \\ \prod_{k=1}^{j-1} \sin\theta_{ik}, & j = d, \quad i = 1,\ldots,n. \end{cases} \quad (2.16)$$

The parameter set θ is defined through $n(d-1)$ angular coordinates θ_{ij}. A reduction of the number of parameters can be done by choosing $\theta_{ij} = \theta_i$, $j = 1,\ldots,d-1$. In this special case, Brigo [5] obtains the following expression

d-rank column-homogeneous angles parameterization

$$\rho_{ij}^{(d)} = \sqrt{\left(1-\alpha_i^2\right)\left(1-\alpha_j^2\right)} \frac{1-\left(\alpha_i \alpha_j\right)^{d-1}}{1-\alpha_i \alpha_j} + \left(\alpha_i \alpha_j\right)^{d-1}, \quad (2.17)$$

$\alpha_k \in (-1,1)$

Here, one has $C^{(d)} = \left(\rho_{ij}^{(d)}\right) = B^{(d)} \cdot B^{(d)T}$, $B = \left(b_{ij}\right)$ with

$$b_{ij} = \left(1-\alpha_i^2\right)^{\frac{1\{j<d\}}{2}} \alpha_i^{j-1}. \quad (2.18)$$

The special case $\alpha_i = \alpha$ for all i yields a correlation matrix with all entries equal to 1. This yields a one-factor LIBOR market model that is sometimes adopted as starting point. The matrix $C(\theta)^{(d)}$ in (2.18) is guaranteed to be a valid correlation matrix. The row vectors of $B(\theta)^{(d)}$ can be viewed as coordinates lying on a unit hyper-sphere. The angles θ_{ij} can be chosen by minimizing the mean square error

$$\left\| C(\theta)^{(d)} - C \right\| = \frac{1}{n} \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\rho(\theta)_{ij}^{(d)} - \rho_{ij}\right)^2} \quad (2.19)$$

To achieve a decrease in the number of free parameters one must have $d < \frac{n-1}{2}$. There are some problems left open in this formulation. For example, in the full rank case one needs $n(n-1)$ angle parameters whereas just half of these parameters are needed to characterize a correlation matrix. Rapirsada et al. [51] solve this problem using a geometric interpretation of the angles parameters.

2.3.3 Correlation Matrix Completion Problems

Consider the following $2n$-dimensional system of stochastic differential equations

$$\begin{aligned} dS_i &= \mu_i S_i dt + f_i(V_i) S_i dW_{(S,i)}, \\ dV_j &= a_j(V_j) dt + b_j(V_j) dW_{(V,j)}, \\ dW_{(S,i)} \cdot dW_{(S,j)}^T &= \rho_{ij} dt, \quad dW_{(S,i)} \cdot dW_{(V,i)} = \eta_i dt, \end{aligned} \quad (2.20)$$

with $i, j = 1,\ldots,n$, and Brownian motions $W_{(S,i)}, W_{(V,j)}$. Since in (2.20) the focus is just on one volatility-underlying pair, one obtains a typical stochastic volatility model to describe the non-flat implied volatility surface in European option's prices (e.g. Heston [23], Chesney and Scott [9] and Schöbel and Zhu [58].

The correlation matrix of the system (2.20) has the following structure

$$C = \left(c_{ij}\right)_{1 \le i,j \le 2n} = \left\{ \begin{pmatrix} \rho_{11} & \cdots & \rho_{1n} & \eta_1 & & ? \\ \cdots & & \cdots & & \cdots & \\ \rho_{n1} & \cdots & \rho_{nn} & ? & & \eta_n \\ \eta_1 & & ? & 1 & & ? \\ & \cdots & & & & \\ ? & & \eta_n & ? & & 1 \end{pmatrix} \right. . \quad (2.21)$$

with undefined coefficients marked "?". The *correlation matrix completion problem* (2.21) is discussed and solved in Kahl and Günther [36], Kahl [35], and Jäckel and Kahl [32]. A general approach to completion problems is in Kurowicka and Cooke [38], Section 4.5.3.

2.4 Algorithmic Validation and Generation of Valid Correlation Matrices

In case one is not interested in the determination of the eigenvalues and the eigenvectors of a correlation matrix, it is not necessary to apply the QR-algorithm from Section 2.1 to ensure a given C is a positive semi-definite matrix. Alternatively, there is a very simple test, which can highly be recommended for software implementation. In fact, there is a simple product representation of the determinant of a correlation matrix in terms of its partial correlation coefficients, which are defined by the recursive formulas (e.g. Yule and Kendall [64]):

$$\rho_{12;3,...,n} = \frac{\rho_{12;3,...,n-1} - \rho_{1n;3,...,n-1} \cdot \rho_{2n;3,...,n-1}}{\sqrt{(1-\rho^2_{1n;3,...,n-1})\cdot(1-\rho^2_{2n;3,...,n-1})}}. \quad (2.22)$$

The determinant of a correlation matrix is given by (e.g. author [26], formula (2.10))

$$\begin{aligned}\det(C) &= \prod_{i=1}^{n-1}\left(1-\rho_{in}^2\right)\cdot\prod_{i=1}^{n-2}\left(1-\rho_{in-1;n}^2\right)\\ &\cdot\prod_{i=1}^{n-3}\left(1-\rho_{in-2;n-1,n}^2\right)\cdot\prod_{k=3}^{n-2}\left\{\prod_{i=1}^{n-k-1}\left(1-\rho_{in-k;n-k+1,...,n}^2\right)\right\}\end{aligned} \quad (2.23)$$

Now, this determinant is non-negative if, and only if, all the partial correlation coefficients in (2.22) belong to the interval $[-1,1]$ (e.g. author [26], Lemma 2.1), i.e.

$$\begin{aligned}&\rho_{in} \in [-1,1], \quad i=1,...,n-1, \quad n\ge 2,\\ &\rho_{in-1;n} \in [-1,1], \quad i=1,...,n-2, \quad n\ge 3,\\ &\rho_{in-2;n-1,n} \in [-1,1], \quad i=1,...,n-3, \quad n\ge 4, \quad (2.24)\\ &\rho_{in-k;n-k+1,...,n} \in [-1,1], \quad i=1,...,n-k-1,\\ &k=3,...,n-2, \quad n\ge 5.\end{aligned}$$

On the other hand, the algorithmic generation of valid correlation matrices has been up to quite recently a challenging problem. A first constructive recursive optimization algorithm for this is described in Budden et al. [6]. An important improvement is author [26], Corollary 4.1, who obtains recursive iterative explicit closed form bounds for the correlation coefficients. Since they do not require (a priori) any optimization procedure these bounds offer a more straightforward implementation than the optimization algorithm by Budden et al. [6]. A general explicit parameterization has finally been obtained in author [29].

3. Single Currency Multifactor LMM

After having justified the need for multifactor LIBOR market models in the introductory Section 1, we provide in the present and next Sections specifications of some single and multiple currency multifactor LIBOR market models.

One assumes a time discretization $0 = T_0 < T_1 < ... < T_n$ (tenor structure), and set the LIBOR forward rates equal to $L_k = \{P(T_k) - P(T_{k+1})\}/\{(T_{k+1}-T_k)P(T_k)\}$, $k = 0,...,n-1$, with $P(T)$ the price of a zero-coupond bond. Let $C = (\rho_{ij}), i,j = 1,...,n$, be a valid correlation matrix for the LIBOR forward rates and consider a suitable d-rank nxd matrix B such that $C = B \cdot B^T$, with $d \le n$. The forward rates of a d-factor LMM in the rolling forward risk-neutral world follow the stochastic differential equations (Fries [17], equation (17.2) and Section 17.1.1.2)

$$\frac{dL_k(t)}{L_k(t)} = \sum_{j=\eta(t)}^{k} \frac{\gamma_k(t)\cdot\gamma_j(t)^T\delta_j L_j(t)}{1+\delta_j L_j(t)}dt + \gamma_k(t)\cdot dZ(t), \quad (3.1)$$
$$k=\eta(t),...,n, \quad \eta(t) = \max\{j : T_j \le t\}+1,$$

$$\begin{aligned}&\gamma_i(t) = \sigma_i(t)\cdot(b_{i1},...,b_{id}), \quad i=1,...,n,\\ &dZ(t) = (dZ_1(t),...,dZ_d(t))^T,\end{aligned} \quad (3.2)$$

where $Z_1(t),...,Z_d(t)$ are independent standard Wiener processes. Let us look explicitly at some low factor models.

One-factor LMM

The case $d=1$ corresponds to the main equation

$$\frac{dL_k(t)}{L_k(t)} = \sum_{j=\eta(t)}^{k} \frac{\sigma_k(t)\sigma_j(t)\delta_j L_j(t)}{1+\delta_j L_j(t)}dt + \sigma_k(t)\cdot dZ(t), \quad (3.3)$$
$$k=\eta(t),...,n.$$

Two-factor LMM

The $nx2$ matrix B can be chosen such that (e.g. Theorem 3 in Rebonato [53])

$$b_{i1} = \cos\theta_i, b_{i2} = \sin\theta_i, b_{ij} = 0, \quad i=1,...,n, \ j=3,...,n. \quad (3.4)$$

With $C = B\cdot B^T$ one obtains the correlation coefficients as

$$\rho_{ij} = b_{i1}b_{j1} + b_{i2}b_{j2} = \cos(\theta_i - \theta_j), \quad i,j=1,...,n. \quad (3.5)$$

Schoenmakers and Coffey [60], Section 2.1, use (3.5) to illustrate that low factor models have intrinsic problems to match (instantaneous) correlations between forward LIBOR rates realistically (see also Rebonato [52], Schoenmakers and Coffey [59]).

Remarks 3.1. It is remarkable that this specification coincides with the extreme points of rank two in the compact and convex set of all correlation matrices. In fact, the pair (3.4)-(3.5) yields a solution of the rank two characterizing equations in Ycart [63], Theorem 2 (see Section 7 for further discussion).

Three-factor LMM

Using the hyper-sphere decomposition (1.16), the $nx3$ matrix B such that $C = B\cdot B^T$ can be chosen such that

$$\begin{aligned}&b_{i1} = \cos\theta_{i1}, \quad b_{i2} = \cos\theta_{i2}\cdot\sin\theta_{i1},\\ &b_{i3} = \sin\theta_{i1}\cdot\sin\theta_{i2}, \quad b_{ij} = 0, \quad i=1,...,n, \ j=4,...,n.\end{aligned} \quad (3.6)$$

This model can be fitted to a “market” correlation structure $C^m = (\rho^m_{ij})$ such that the mean square error $\|C^m - C\|$ in

(1.19) is minimized. For example, Rebonato [53] fits successfully a correlation structure of type (1.12) given by

$$\begin{aligned}\rho_{ij} &= \rho_\infty + (1-\rho_\infty) \\ &\cdot \exp\{-|T_i - T_j| \cdot (\beta + \alpha \cdot \max(T_i, T_j))\}, \quad (3.7) \\ \rho_\infty &= 0.3,\ \beta = 0.12,\ \alpha = 0.005\end{aligned}$$

Alexander and Lvov [2] and Lim [41] report excellent results when fitting the correlation surface with a three-factor LMM. A recent master thesis on this topic is Tamminen [62].

4. Multiple Currency Multifactor LMM

Following Fries [17], Chapter 24, and Kawai and Jäckel [37], we expand now the specification to a multiple currency multifactor LIBOR market model.

4.1 Cross currency multifactor models

Consider the case of a two currencies LMM with d stochastic factors, each for the domestic and foreign LIBOR forward rates, and one factor for the foreign/domestic spot exchange rate. The domestic currency is denoted by DDD and the foreign currency by FFF. The spot exchange rate FX refers to $FX = FFF/DDD$, i.e. the number of units of domestic currency that equal one unit of foreign currency. We assume that for the chosen numéraire that there exists a corresponding equivalent martingale measure under which the modelled quantities are log-normal processes.

4.1.1 Full-Rank Models

We begin with the full-rank model with $d = n$ Wiener processes for the domestic and foreign forward rates and one Wiener process for the spot FX rate. With the usual definitions and notations, one has the stochastic differential equations

Domestic LIBOR forward rates

$$\frac{dL_i^D(t)}{L_i^D(t)} = \mu_i^D(t)dt + \sigma_i^D(t)dW_i^D(t), \quad i = 1,...,n \quad (4.1)$$

Foreign LIBOR forward rates

$$\frac{dL_i^F(t)}{L_i^F(t)} = \mu_i^F(t)dt + \sigma_i^F(t)dW_i^F(t), \quad i = 1,...,n \quad (4.2)$$

Spot exchange rate

$$\frac{dFX(t)}{FX(t)} = \mu^{FX}(t)dt + \sigma^{FX}(t)dW^{FX}(t) \quad (4.3)$$

Case 1: Drift terms under the domestic rolling forward risk measure or spot LIBOR measure

$$\mu_i^D(t) = \sigma_i^D(t) \cdot \sum_{j=\eta(t)}^{i} \frac{\rho_{ij}^{(D,D)}\sigma_j^D(t)\delta_j^D L_j^D(t)}{1+\delta_j^D L_j^D(t)}, \quad (4.4)$$

$$\rho_{ij}^{(D,D)}dt = E\left[dW_i^D(t)dW_j^D(t)\right]$$

$$\mu_i^F(t) = \sigma_i^F(t) \cdot \left[\begin{array}{l} \sum_{j=\eta(t)}^{i} \frac{\rho_{ij}^{(F,F)}\sigma_j^F(t)\delta_j^F L_j^F(t)}{1+\delta_j^F L_j^F(t)} \\ -\rho_i^{(F,FX)}\sigma^{FX}(t) \end{array}\right],$$

$$\rho_{ij}^{(F,F)}dt = E\left[dW_i^F(t)dW_j^F(t)\right], \quad (4.5)$$

$$\rho_i^{(F,FX)}dt = E\left[dW_i^F(t)dW^{FX}(t)\right]$$

$$\mu^{FX}(t) = \frac{\partial}{\partial t}\ln\left\{\frac{P^D(t;T_{\eta(t)})}{P^F(t;T_{\eta(t)})}\right\} \quad (4.6)$$

A mathematical derivation of these formulas is found in Fries [17], Section 24.1.1.

Case 2: Drift terms under the domestic terminal measure

$$\mu_i^D(t) = -\sigma_i^D(t) \cdot \sum_{j=i+1}^{n} \frac{\rho_{ij}^{(D,D)}\sigma_j^D(t)\delta_j^D L_j^D(t)}{1+\delta_j^D L_j^D(t)} \quad (4.7)$$

$$\mu_i^F(t) = \sigma_i^F(t) \cdot \left[\begin{array}{l} \sum_{j=1}^{i} \frac{\rho_{ij}^{(F,F)}\sigma_j^F(t)\delta_j^F L_j^F(t)}{1+\delta_j^F L_j^F(t)} \\ -\rho_i^{(F,FX)}\sigma^{FX}(t) \\ -\sum_{j=1}^{n} \frac{\rho_{ij}^{(D,F)}\sigma_j^D(t)\delta_j^D L_j^D(t)}{1+\delta_j^D L_j^D(t)} \end{array}\right], \quad (4.8)$$

$$\rho_{ij}^{(D,F)}dt = E\left[dW_i^D(t)dW_j^F(t)\right]$$

$$\begin{aligned}\mu^{FX}(t) &= r^D(t) - r^F(t) \\ &-\sigma^{FX}(t) \cdot \sum_{j=1}^{n} \frac{\rho_j^{(D,FX)}\sigma_j^D(t)\delta_j^D L_j^D(t)}{1+\delta_j^D L_j^D(t)}, \quad (4.9)\end{aligned}$$

$$\rho_j^{(D,FX)}dt = E\left[dW_j^D(t)dW^{FX}(t)\right]$$

These formulas and their interpretations are found in Kawai and Jäckel [37], Section 2.

Remark 4.1. One notes that the drifts in Case 1 do not depend on the cross currency correlations $\rho_{ij}^{(D,F)}$ and on the domestic currency to FX correlations $\rho_i^{(D,FX)}$.

For several purposes, in particular Monte Carlo simulation, it is more convenient to express the stochastic differential equations (4.1)-(4.3) as being driven by a vector of $2n+1$ independent standard Wiener processes $Z(t) = (Z_1(t),...,Z_{2n+1}(t))^T$ to obtain a formulation of the type shown in (3.1) and (3.2). For this, one considers the $(2n+1)x(2n+1)$-matrix composed of the inter currency domestic and foreign correlations $\rho_{ij}^{(D,D)}, \rho_{ij}^{(F,F)}$, the cross currency correlations $\rho_{ij}^{(D,F)}$, and the currency to FX correlations $\rho_i^{(D,FX)}, \rho_i^{(F,FX)}$, defined by

$$C=\begin{pmatrix}\rho_{ij}^{(D,D)} & \rho_{ij}^{(D,F)} & \rho_{i}^{(D,FX)}\\ \rho_{ij}^{(D,F)} & \rho_{ij}^{(F,F)} & \rho_{i}^{(F,FX)}\\ \rho_{j}^{(D,FX)} & \rho_{j}^{(F,FX)} & 1\end{pmatrix}. \quad (4.10)$$

Let (4.10) be a valid correlation matrix. Consider its spectral decomposition $C = B\cdot B^T$ and define the $(2n+1)$-dimensional volatility vectors

$$\begin{aligned}&\gamma_i^D(t)=\sigma_i^D(t)\cdot(b_{i1},...,b_{i2n+1}), \quad i=1,...,n,\\ &\gamma_i^F(t)=\sigma_i^F(t)\cdot(b_{n+i1},...,b_{n+i2n+1}), \quad i=1,...,n, \quad (4.11)\\ &\gamma^{FX}(t)=\sigma^{FX}(t)\cdot(b_{2n+11},...,b_{2n+12n+1}),\end{aligned}$$

which satisfy the following precise relationships

$$\begin{aligned}&\gamma_i^D(t)\cdot\gamma_j^D(t)^T=\sigma_i^D(t)\sigma_j^D(t)\rho_{ij}^{(D,D)},\\ &\gamma_i^F(t)\cdot\gamma_j^F(t)^T=\sigma_i^F(t)\sigma_j^F(t)\rho_{ij}^{(F,F)},\\ &\gamma_i^D(t)\cdot\gamma_j^F(t)^T=\sigma_i^D(t)\sigma_j^F(t)\rho_{ij}^{(D,F)},\\ &\gamma^{FX}(t)\cdot\gamma^{FX}(t)^T=\sigma^{FX}(t)^2,\\ &\gamma_i^D(t)\cdot\gamma^{FX}(t)^T=\sigma_i^D(t)\sigma^{FX}(t)\rho_i^{(D,FX)},\\ &\gamma_j^F(t)\cdot\gamma^{FX}(t)^T=\sigma_j^F(t)\sigma^{FX}(t)\rho_j^{(F,FX)}.\end{aligned} \quad (4.12)$$

Under this orthogonal transformation the stochastic differential equations (4.1)-(4.3) read

$$\frac{dL_i^D(t)}{L_i^D(t)}=\mu_i^D(t)dt+\gamma_i^D(t)dZ(t), \quad i=1,...,n \quad (4.13)$$

$$\frac{dL_i^F(t)}{L_i^F(t)}=\mu_i^F(t)dt+\gamma_i^F(t)dZ(t), \quad i=1,...,n \quad (4.14)$$

$$\frac{dFX(t)}{FX(t)}=\mu^{FX}(t)dt+\gamma^{FX}(t)dZ(t) \quad (4.15)$$

where the drifts can be rewritten in compact form as follows.

Case 1:

$$\mu_i^D(t)=\sum_{j=\eta(t)}^{i}\frac{\gamma_i^D(t)\cdot\gamma_j^D(t)^T\delta_j^D L_j^D(t)}{1+\delta_j^D L_j^D(t)} \quad (4.16)$$

$$\mu_i^F(t)=\sum_{j=\eta(t)}^{i}\frac{\gamma_i^F(t)\cdot\gamma_j^F(t)^T\delta_j^F L_j^F(t)}{1+\delta_j^F L_j^F(t)} -\gamma_i^F(t)\cdot\gamma^{FX}(t)^T \quad (4.17)$$

$$\mu^{FX}(t)=\frac{\partial}{\partial t}\ln\left\{\frac{P^D(t;T_{\eta(t)})}{P^F(t;T_{\eta(t)})}\right\} \quad (4.18)$$

Case 2:

$$\mu_i^D(t)=-\sum_{j=i+1}^{n}\frac{\gamma_i^D(t)\cdot\gamma_j^D(t)^T\delta_j^D L_j^D(t)}{1+\delta_j^D L_j^D(t)} \quad (4.19)$$

$$\mu_i^F(t)=\sum_{j=1}^{i}\frac{\gamma_i^F(t)\cdot\gamma_j^F(t)^T\delta_j^F L_j^F(t)}{1+\delta_j^F L_j^F(t)} -\gamma_i^F(t)\cdot\gamma^{FX}(t)^T-\sum_{j=1}^{n}\frac{\gamma_i^F(t)\cdot\gamma_j^D(t)^T\delta_j^D L_j^D(t)}{1+\delta_j^D L_j^D(t)} \quad (4.20)$$

$$\mu^{FX}(t)=r^D(t)-r^F(t)-\sum_{j=1}^{n}\frac{\gamma_j^D(t)\cdot\gamma^{FX}(t)^T\delta_j^D L_j^D(t)}{1+\delta_j^D L_j^D(t)} \quad (4.21)$$

4.1.2 Reduced-rank models

Given are n_D domestic and n_F foreign forward rates and set $n=n_D+n_F+1$. At this stage it is easy to state the reduced-rank model with $1\le d\le\min(n_D,n_F)$ Wiener processes for the domestic and foreign forward rates and one Wiener process for the spot FX rate. It consists of the equations (4.13)-(4.15), which are driven by a vector of $2d+1$ independent standard Wiener processes $Z(t)=(Z_1(t),...,Z_{2d+1}(t))^T$. The drifts (4.16)-(4.18) in Case 1 and (4.19)-(4.21) in Case 2 are defined in terms of $(2d+1)$-dimensional volatility vectors

$$\begin{aligned}&\gamma_i^D(t)=\sigma_i^D(t)\cdot(b_{i1},...,b_{i2d+1}), \quad i=1,...,n_D,\\ &\gamma_i^F(t)=\sigma_i^F(t)\cdot(b_{n_D+i1},...,b_{n_D+i2d+1}), \quad i=1,...,n_F, \quad (4.22)\\ &\gamma^{FX}(t)=\sigma^{FX}(t)\cdot(b_{n1},...,b_{n2d+1}),\end{aligned}$$

satisfying (4.12). In other words, the $(2d+1)$-rank $nx(2d+1)$ matrix B fulfils by construction the reduced-rank spectral decomposition $C=B\cdot B^T$. We note that the special case $d=n_D=n_F$ coincides with the full-rank model of Section 4.1.1.

4.2 Multiple Currency Multifactor Models With Stock and Commodities Indices (SCI)

The model of Section 4.1.2 for the Case 1 of the domestic rolling forward risk measure is now generalized to the case of more than one foreign currency in the presence of non-dividend paying stock indices. Our generalization uses results from Fries [17], Section 24.3.

We denote the domestic currency by $CUR_0 = DDD$, the foreign currencies by CUR_j, $j=1,...,n_C$, and the spot exchange rates by $FX^j = CUR_j / CUR_0$. For the j^{th} currency there are given n_j LIBOR forward rates and n_j^S non-dividend paying stock and commodity (SCI) indices. The i^{th} LIBOR forward rate of the j^{th} currency is denoted by $L_i^j(t)$, the j^{th} spot exchange rate by $FX^j(t)$, and the i^{th} non-dividend paying SCI of the j^{th} currency is denoted by $S_i^j(t)$. The number defined by $n=n_C+\sum_{j=0}^{n_C}(n_j+n_j^S)$ is equal to the dimension of the correlation matrix C between the Itô processes associated to the stochastic variables $L_i^j(t)$, $FX^j(t)$ and $S_i^j(t)$. We consider the case of a multicurrency LIBOR market model with $1\le d\le\min(n_0,n_1,...,n_{n_C})$ stochastic factors, each for the domestic and foreign LIBOR forward rates, and one stochastic factor, each for the foreign/domestic spot exchange rates and the non-dividend paying SCI's. With the appropriate definitions and notations, the multiple currency multifactor LMM can be summarized by the following stochastic differential equations, which are driven by a vector of $N:=(n_C+1)d+n_C+\sum_{j=0}^{n_C}n_j^S$

independent standard Wiener processes $Z(t)=(Z_1(t),...,Z_N(t))^T$.

LIBOR forward rates

$$\frac{dL_i^j(t)}{L_i^j(t)}=\mu_i^j(t)dt+\gamma_i^j(t)dZ(t), \quad i=1,...,n_j,\ j=0,1,...,n_C \tag{4.23}$$

Spot exchange rates

$$\frac{dFX^j(t)}{FX^j(t)}=\mu_{FX}^j(t)dt+\gamma_{FX}^j(t)dZ(t), \quad j=1,...,n_C \tag{4.24}$$

Non-dividend paying SCI's

$$\frac{dS_i^j(t)}{S_i^j(t)}=\mu_{S,i}^j(t)dt+\gamma_{S,i}^j(t)dZ(t), \quad i=1,...,n_j^S,\ j=0,1,...,n_C \tag{4.25}$$

It remains to specify in these equations the N-dimensional volatility vectors and the drift terms. For convenience we consider the following index functions

$$\alpha(j)=\begin{cases}0, & j=0,\\ \sum_{k=0}^{j-1}n_k, & j=1,...,n_C+1,\end{cases} \tag{4.26}$$

$$\beta(j)=\alpha(n_C+1)+n_C+\begin{cases}0, & j=0,\\ \sum_{k=0}^{j-1}n_k^S, & j=1,...,n_C+1,\end{cases} \tag{4.27}$$

where by construction $\beta(n_C+1)=n$ is the dimension of the correlation matrix C. Then, the volatility vectors in (4.23)-(4.25), which are parameterized by

$$\begin{aligned}&\gamma_i^j(t)=\sigma_i^j(t)\cdot(b_{\alpha(j)+i,1},...,b_{\alpha(j)+i,N}),\\ &i=1,...,n_j,\ j=0,1,...,n_C,\\ &\gamma_{FX}^j(t)=\sigma_{FX}^j(t)\cdot(b_{\alpha(n_C+1)+j,1},...,b_{\alpha(n_C+1)+j,N}),\\ &j=1,...,n_C,\\ &\gamma_{S,i}^j(t)=\sigma_{S,i}^j(t)\cdot(b_{\beta(j)+i,1},...,b_{\beta(j)+i,N}),\\ &i=1,...,n_j^S,\ j=0,1,...,n_C,\end{aligned} \tag{4.28}$$

must (by construction of the decomposition $C=B\cdot B^T$) satisfy the covariance relationships

$$\begin{aligned}&\gamma_i^j(t)\cdot\gamma_k^\ell(t)^T=\sigma_i^j(t)\sigma_k^\ell(t)\rho_{ik}^{(j,\ell)},\\ &i=1,...,n_j,\ j=0,1,...,n_C,\ k=1,...,n_\ell,\ \ell=0,1,...,n_C,\\ &\gamma_{FX}^j(t)\cdot\gamma_{FX}^\ell(t)^T=\sigma_{FX}^j(t)\sigma_{FX}^\ell(t)\rho_{FX}^{(j,\ell)},\\ &j=1,...,n_C,\ \ell=1,...,n_C,\\ &\gamma_{S,i}^j(t)\cdot\gamma_{S,k}^\ell(t)^T=\sigma_{S,i}^j(t)\sigma_{S,k}^\ell(t)\rho_{S,ik}^{(j,\ell)},\\ &i=1,...,n_j^S,\ j=0,1,...,n_C,\ k=1,...,n_\ell^S,\ \ell=0,1,...,n_C,\\ &\gamma_i^j(t)\cdot\gamma_{FX}^\ell(t)^T=\sigma_i^j(t)\sigma_{FX}^\ell(t)\rho_{FX,i}^{(j,\ell)},\\ &i=1,...,n_j,\ j=0,1,...,n_C,\ \ell=1,...,n_C,\\ &\gamma_i^j(t)\cdot\gamma_{S,k}^\ell(t)^T=\sigma_i^j(t)\sigma_{S,k}^\ell(t)\rho_{S,ik}^{(j,\ell)},\\ &i=1,...,n_j,\ j=0,1,...,n_C,\ k=1,...,n_\ell^S,\ \ell=0,1,...,n_C,\\ &\gamma_{S,i}^j(t)\cdot\gamma_{FX}^\ell(t)^T=\sigma_{S,i}^j(t)\sigma_{FX}^\ell(t)\rho_{(S,FX),i}^{(j,\ell)},\\ &i=1,...,n_j^S,\ j=0,1,...,n_C,\ \ell=1,...,n_C.\end{aligned} \tag{4.29}$$

The drifts in (4.23)-(4.25) are given by

$$\begin{aligned}\mu_i^j(t)=&\sum_{k=\eta(t)}^{i}\frac{\gamma_i^j(t)\cdot\gamma_k^j(t)^T\delta_k^jL_k^j(t)}{1+\delta_k^jL_k^j(t)}\\ &-\gamma_i^j(t)\cdot\gamma_{FX}^j(t)^T\cdot 1\{j>0\},\\ &i=1,...,n_j,\quad j=0,1,...,n_C,\end{aligned} \tag{4.30}$$

$$\mu_{FX}^j(t)=\frac{\partial}{\partial t}\ln\left\{\frac{P^0(t;T_{\eta(t)})}{P^j(t;T_{\eta(t)})}\right\},\quad j=1,...,n_C, \tag{4.31}$$

$$\mu_{S,i}^j(t)=\frac{\partial}{\partial t}\ln\{P^j(t;T_{\eta(t)})\}-\gamma_{S,i}^j(t)\cdot\gamma_{FX}^j(t)^T\cdot 1\{j>0\}, \quad i=1,...,n_j^S,\quad j=0,1,...,n_C, \tag{4.32}$$

where $P^j(t;T_{\eta(t)})$ are zero bond prices in the j^{th} currency, $j=0,1,...,n_C$. Furthermore, if one chooses for $P^j(t;T_{\eta(t)})$ the interpolation defined by

$$P^j(t;T_{i+1}):=(1+L_i^j(T_i)\cdot(T_{i+1}-t))^{-1},\quad T_i<t<T_{i+1}, \tag{4.33}$$

then the drift terms in (4.31)-(4.32) simplify to

$$\mu_{FX}^j(t)=\frac{\partial}{\partial t}\ln\left\{\frac{1+L_{\eta(t)-1}^j(T_{\eta(t)-1})\cdot(T_{\eta(t)}-t)}{1+L_{\eta(t)-1}^0(T_{\eta(t)-1})\cdot(T_{\eta(t)}-t)}\right\},\quad j=1,...,n_C, \tag{4.34}$$

$$\begin{aligned}\mu_{S,i}^j(t)=&\frac{\partial}{\partial t}\ln\left\{\frac{1}{1+L_{\eta(t)-1}^j(T_{\eta(t)-1})\cdot(T_{\eta(t)}-t)}\right\}\\ &-\gamma_{S,i}^j(t)\cdot\gamma_{FX}^j(t)^T\cdot 1\{j>0\},\\ &i=1,...,n_j^S,\quad j=0,1,...,n_C.\end{aligned} \tag{4.35}$$

We note that the N-rank nxN matrix B in (4.28) fulfils by the construction (4.29) the reduced-rank spectral decomposition $C=B\cdot B^T$. It is instructive to look explicitly at the following known special cases.

Single currency multifactor LMM

The choice $n_C=0,\ (n_0,n_0^S)=(n,0),\ 1\le d\le n,$ yields the d-factor single currency LMM defined by (3.1)-(3.2).

Cross currency multifactor LMM

The choice

$$n_C=1,\ (n_0,n_1,n_0^S,n_1^S)=(n_D,n_F,0,0),\ 1\le d\le \min(n_D,n_F),$$

yields the d-factor cross currency LMM with Case 1 drifts in Section 4.1.2, and Fries [17], Section 24.1.

Equity hybrid multifactor LMM

The choice $n_C = 0, (n_0, n_0^S) = (n,1), 1 \le d \le n,$ yields the d-factor single currency equity hybrid LMM considered earlier in Fries [17], Section 24.2.

Cross currency equity hybrid multifactor LMM

The choice

$$n_C = 1, (n_0, n_1, n_0^S, n_1^S) = (n_D, n_F, 1, 1), 1 \le d \le \min(n_D, n_F),$$

yields the d-factor cross currency equity hybrid LMM considered earlier in Fries(2007), Section 24.3.

5. Greeks for Multiple Currency Multifactor LMM

We are interest in simulation methods that can be designed for the case of discontinuous payoffs and even arbitrary path-dependent payoffs. In a test case, we have successively applied the LRM algorithm of Glasserman and Zhao [20] to the calculation of deltas for digital and knockout caplet (cap) interest rate options within the framework of the single currency full-rank LMM. In the present Section, we propose to extend the use of the LRM algorithm to arbitrary path-dependent payoffs for the multiple currency multifactor LIBOR market model.

Fixing a currency and omitting the corresponding upper indices, the dynamics of the k^{th} LIBOR forward rate in the domestic currency can be expressed in the form

$$\frac{dL_k(t)}{L_k(t)} = \mu_k(t)dt + \gamma_k(t)dZ(t), \quad k = \eta(t), \ldots, n \,. \quad (5.1)$$

In the following, the choice of a tenor structure with constant $\delta_i = T_{i+1} - T_i = \delta$ and MC simulation step $h = \delta$ is made. In a next step, the assumption $h = \delta$ must be validated or rejected by testing the accuracy of the LMM against various applications. In this simplified framework, the Euler scheme of the logarithmic process in (5.1) is given by (here under the convention $T_0 = 0 < T_1 = \delta < \ldots < T_i = \delta \cdot i < \ldots < T_n = n \cdot \delta$)

$$\begin{aligned}
&\ln L_k(T_{i+1}) = \ln L_k(T_i) + \delta \cdot \{\mu_k(T_i) - \tfrac{1}{2}\|\gamma_k(T_i)\|^2\} \\
&+ \sqrt{\delta} \cdot \gamma_k(T_i) \cdot Z_{i+1}, \quad k = T_{i+1}, \ldots, n \\
&\ln L_k(T_{i+1}) = \ln L_k(T_i), \quad k < T_{i+1}, \\
&k = 1, \ldots, n, \quad i = 0,1,2,\ldots, n-1,
\end{aligned} \quad (5.2)$$

where the Z_i's are independent N-dimensional standard normal vectors and from (4.30)

$$\begin{aligned}
\mu_k(T_i) &= \sum_{r=i+1}^{k} \frac{\gamma_k(T_i) \cdot \gamma_r(T_i)^T \delta L_r(T_i)}{1 + \delta L_r(T_i)} \\
&- \gamma_k(T_i) \cdot \gamma_{FX}(T_i)^T \cdot 1\{foreign\ currency\}
\end{aligned} \quad . (5.3)$$

Over a single time step, the covariance matrix of increments in (5.2) has rank $N \le n$. In case $N < n$ this matrix is singular. One considers the distribution of the increments over multiple time steps by rewriting (5.2) as

$$\begin{aligned}
&\ln L_k(T_i) = \ln L_k(T_0) + \delta \cdot \sum_{j=0}^{i-1} \left(\mu_k(T_j) - \tfrac{1}{2}\|\gamma_k(T_j)\|^2 \right) \\
&+ \sqrt{\delta} \cdot (\gamma_k(T_0), \gamma_k(T_1), \ldots, \gamma_k(T_{i-1})) \cdot (Z_1, Z_2, \ldots, Z_i)^T, \quad (5.4) \\
&k = 1, \ldots, n, \quad i = 1,2,\ldots, n,
\end{aligned}$$

where the row vectors $\gamma_k(T_j)$ have been concatenated into a single vector of length $i \cdot N$ and the column vectors Z_j have been stacked into a column of the same length. Depending on the type of multiple currency multifactor LMM, it is possible to find an integer $J \ge 1$ such that the $nxJ \cdot N$ matrix

$$\Lambda(J) = \begin{pmatrix} \gamma_1(T_0) & \gamma_1(T_1) & \ldots & \gamma_1(T_{J-1}) \\ \gamma_2(T_0) & \gamma_2(T_1) & \ldots & \gamma_2(T_{J-1}) \\ \ldots & \ldots & \ldots & \ldots \\ \gamma_n(T_0) & \gamma_n(T_1) & \ldots & \gamma_n(T_{J-1}) \end{pmatrix} \quad (5.5)$$

is assumed to have rank n, even if $N < n$ ("mild" regularity model assumption). In this case the matrix $\Lambda(J) \cdot \Lambda(J)^T$ is invertible and the LRM method in [20], Section 4.2, can be applied. Suppose that $\Lambda(J)$ has full rank. To calculate the delta of a payoff function $g(x)$ with respect to the LIBOR forward rate $\theta = L_k(T_0)$, we apply the equality $\frac{\partial}{\partial \theta} E[g(X)] = E\left[g(X)(X - \mu(\theta))^T \Sigma^{-1} \dot{\mu}(\theta)\right]$, valid in a Gaussian setting by making the following correspondences

$$X = (X_1, \ldots, X_n) \leftarrow (\ln L_1(T_J), \ldots, \ln L_n(T_J)),$$

$\mu(\theta) = (\mu_1(\theta), \ldots, \mu_n(\theta))$ with

$$\mu_i(\theta) = \ln L_i(T_0) + \delta \cdot \sum_{j=0}^{J-1} \{\mu_i^0(T_j) - \tfrac{1}{2}\|\gamma_i(T_j)\|^2\},$$

$i = 1, \ldots, n,$

where $\mu_i^0(\cdot)$ is the *forward-drift approximation* to (5.3) proposed in [20], Section 3.3, which is defined by

$$\begin{aligned}
\mu_i^0(T_j) &= \sum_{r=j+1}^{i} \frac{\gamma_i(T_j) \cdot \gamma_r(T_j)^T \delta L_r(T_0)}{1 + \delta L_r(T_0)} \\
&- \gamma_i(T_j) \cdot \gamma_{FX}(T_j)^T \cdot 1\{foreign\ currency\}
\end{aligned}$$

$$\dot{\mu}(\theta) \leftarrow \dot{\mu}^{(k)}(\theta) = (\dot{\mu}_1^{(k)}(\theta), \ldots, \dot{\mu}_n^{(k)}(\theta))$$

with

$$\dot{\mu}_i^{(k)}(\theta) = \frac{1\{i=k\}}{L_k(T_0)} + \delta \cdot \sum_{j=0}^{J-1} \frac{\partial \mu_i^0(T_j)}{\partial L_k(T_0)}$$
$$= \frac{1\{i=k\}}{L_k(T_0)} + \delta \cdot \sum_{j=0}^{J-1} \frac{\gamma_i(T_j) \cdot \gamma_k(T_j)^T \delta}{[1+\delta L_k(T_0)]^2} \cdot 1\{j+1 \le k \le i\},$$
$$i = 1, \dots, n.$$

$$\Sigma \leftarrow \delta \cdot \Lambda(J) \cdot \Lambda(J)^T$$

Let now $g(L_1(T_1), \dots, L_n(T_n))$ be the simulated path-wise value of an arbitrary payoff. Then one has the following *LRM delta path-wise estimator*

$$g(L_1(T_1), \dots, L_n(T_n)) \cdot (X - \mu(\theta))^T \Sigma^{-1} \dot{\mu}(\theta) \quad (5.6)$$

where $X, \mu(\theta), \Sigma, \dot{\mu}(\theta)$ have been defined above. Making use of the Euler scheme (5.4), one sees that

$$(X - \mu(\theta))^T = (\sqrt{\delta} \cdot \sum_{\ell=1}^{J} \gamma_1(T_{\ell-1}) \cdot Z_\ell, \dots, \sqrt{\delta} \cdot \sum_{\ell=1}^{J} \gamma_n(T_{\ell-1}) \cdot Z_\ell) \cdot \quad (5.7)$$

With (5.7) the second expression in (5.6) simplifies to

$$(X - \mu(\theta))^T \Sigma^{-1} \dot{\mu}(\theta) = \sqrt{\delta^{-1}} \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_{ij} \dot{\mu}_j^{(k)}(\theta) \sum_{\ell=1}^{J} \gamma_i(T_{\ell-1}) \cdot Z_\ell \quad , (5.8)$$

where the ξ_{ij}'s are the entries of the inverse matrix of $\Lambda(J) \cdot \Lambda(J)^T$.

6. Application: Currency Option Prices and Deltas

To illustrate, we test the MC simulation of European currency option prices and deltas for the cross currency multifactor LMM of Section 4.1, Case 2, against the asymptotic expansion formulas by Kawai and Jäckel [37].

The payoff of a European FX call option with maturity date T and strike rate $K > 0$ is given by $(S(T) - K)_+$, where $S(t)$ denotes the spot exchange rate at time $t \ge 0$. One observes that the spot exchange rate $S(T)$ can be expressed in terms of a foreign exchange forward (forex forward) rate with the same maturity date. That is one has $S(T) = F_T(T)$, where $F_T(t)$ denotes the time t value of the forex forward rate with maturity date T. We will make use of the arbitrage-free relationship between the forex spot rate and the forex forward rate given by

$$F_T(t) = S(t) \cdot \frac{P^F(t,T)}{P^D(t,T)}, \quad (6.1)$$

where $P^D(t,T)$ and $P^F(t,T)$ denote the time t values of domestic and foreign zero coupon bonds with maturity T respectively. The goal is to determine the call option price at time $t = 0$

$$C_{FX}(0) = P^D(0,T) \cdot E^T[(F_T(T) - K)_+] \quad (6.2)$$

and the price sensitivity or delta

$$\frac{\partial C_{FX}(0)}{\partial S(0)}. \quad (6.3)$$

In (6.2) the notation $E^T[\cdot]$ stands for the expectation with respect to the domestic terminal measure. Therefore, to determine price and delta, the distribution of $F_T(T)$ under the domestic terminal measure must be evaluated. For this, we use the cross currency multifactor LIBOR market model as specified in Section 4.1, Case 2. In this setting, the spot exchange rate satisfies the stochastic differential equation (4.15) with drift (4.21) and volatility vector determined by (4.11). The forex forward rate $F_T(T)$, defined by (6.1), is a martingale measure under the domestic terminal measure. Applying Itô's formula to (6.1) one sees that it satisfies the following stochastic differential equation under this measure

$$\frac{dF_T(t)}{F_T(t)} = \{\gamma^{FX}(t) + \eta(t)\} \cdot dZ(t), \quad (6.4)$$

with $\gamma^{FX}(t)$ as in (4.11) and

$$\eta(t) = \sum_{j=1}^{n} \frac{\delta_j^D L_j^D(t)}{1 + \delta_j^D L_j^D(t)} \cdot \gamma_j^D(t) - \sum_{j=1}^{n} \frac{\delta_j^F L_j^F(t)}{1 + \delta_j^F L_j^F(t)} \cdot \gamma_j^F(t) \quad . (6.5)$$

Consider now the asymptotic expansion in [37]. Insert a perturbation parameter ε into equation (6.4) to get

$$F_T^{(\varepsilon)}(T) = S(0) + \varepsilon \cdot \int_0^T F_T^{(\varepsilon)}(t) \cdot (\gamma^{FX}(t) + \eta^{(\varepsilon)}(t)) dZ(t). \quad (6.6)$$

with

$$\eta^{(\varepsilon)}(t) = \sum_{j=1}^{n} \frac{\delta_j^D L_j^{D(\varepsilon)}(t)}{1 + \delta_j^D L_j^{D(\varepsilon)}(t)} \cdot \gamma_j^D(t) - \sum_{j=1}^{n} \frac{\delta_j^F L_j^{F(\varepsilon)}(t)}{1 + \delta_j^F L_j^{F(\varepsilon)}(t)} \cdot \gamma_j^F(t) \quad , (6.7)$$

where the perturbed LIBOR forward rates satisfy the equations

$$L_i^{D(\varepsilon)}(t) = L_i^D(0) + \varepsilon \cdot \int_0^t L_i^D(0) \mu_i^{D(0)}(u) du + \varepsilon \cdot \int_0^t L_i^{D(\varepsilon)}(u) \gamma_i^D(u) dZ(u), \quad i = 1, \dots, n \quad , (6.8)$$

with

$$\mu_i^{D(0)}(u) = -\sum_{j=i+1}^{n} \frac{\gamma_i^D(u) \cdot \gamma_j^D(u)^T \delta_j^D L_j^D(0)}{1 + \delta_j^D L_j^D(0)}, \quad (6.9)$$

and

$$L_i^{F(\varepsilon)}(t) = L_i^F(0) + \varepsilon \cdot \int_0^t L_i^F(0)\mu_i^{F(0)}(u)du + \varepsilon \cdot \int_0^t L_i^{F(\varepsilon)}(u)\gamma_i^F(u)dZ(u), \quad i = 1,\dots,n \quad (6.10)$$

with

$$\mu_i^{F(0)}(u) = \sum_{j=1}^{i} \frac{\gamma_i^F(u)\cdot\gamma_j^F(u)^T \delta_j^F L_j^F(0)}{1+\delta_j^F L_j^F(0)} - \gamma_i^F(u)\cdot\gamma^{FX}(u)^T - \sum_{j=1}^{n} \frac{\gamma_i^F(u)\cdot\gamma_j^D(u)^T \delta_j^D L_j^D(0)}{1+\delta_j^D L_j^D(0)}. \quad (6.11)$$

One notes that the drift terms of the forward rates are approximated deterministically using initial forward rates as a consequence of the Itô-Taylor expansions in (6.6), (6.8) and (6.10). This is similar to the forward-drift approximation already applied in Section 5 and originally proposed by Glasserman and Zhao [20], Section 3.3. It makes the derivation and the resulting formulas simpler while they remain accurate. Applying a Taylor series expansion in ε to the forex forward rate (6.6), one obtains asymptotic expansions of order $k = 1,2,3$ defined by

$$F_T^{(\varepsilon)}(T) = S(0) + \sum_{j=1}^{k} \frac{\varepsilon^j}{j!} \cdot \frac{\partial^{(j)} F_T^{(\varepsilon)}(T)}{\partial \varepsilon^{(j)}}\Big|_{\varepsilon=0}. \quad (6.12)$$

<u>First-order asymptotic expansion</u>

A straightforward calculation shows that

$$\frac{\partial F_T^{(\varepsilon)}(T)}{\partial \varepsilon}\Big|_{\varepsilon=0} = S(0)\cdot\int_0^T \left(\gamma^{FX}(t) + \eta^{(0)}(t)\right)dZ(t), \quad (6.13)$$

from which it follows that

$$X^{(\varepsilon)}(T) = \frac{1}{\varepsilon}\left(F_T^{(\varepsilon)}(T) - S(0)\right) = \frac{\partial F_T^{(\varepsilon)}(T)}{\partial \varepsilon}\Big|_{\varepsilon=0} \sim N(0, S(0)^2\sigma_1^2 T) \quad (6.14)$$

is normally distributed with mean zero and variance $S(0)^2\sigma_1^2 T$ such that

$$\sigma_1^2 = \frac{1}{T}\cdot\left\| \int_0^T \left(\gamma^{FX}(t) + \eta^{(0)}(t)\right)dZ(t)\right\|^2. \quad (6.15)$$

To provide the detail of the latter constant, it is useful to introduce the following auxiliary quantities. Define the time t weights of the initial LIBOR forward rates by

$$w_j^D(t) = \frac{\delta_j^D L_j^D(t)}{1+\delta_j^D L_j^D(t)}, \quad w_j^F(t) = \frac{\delta_j^F L_j^F(t)}{1+\delta_j^F L_j^F(t)}, \quad j = 1,2,3 \quad (6.16)$$

and the integrated terminal covariances between the stochastic processes by

$$\begin{aligned} c_{ij}^{(D,D)} &= \int_0^T \sigma_i^D(t)\sigma_j^D(t)\rho_{ij}^{(D,D)}dt, \\ c_{ij}^{(F,F)} &= \int_0^T \sigma_i^F(t)\sigma_j^F(t)\rho_{ij}^{(F,F)}dt, \\ c_{ij}^{(D,F)} &= \int_0^T \sigma_i^D(t)\sigma_j^F(t)\rho_{ij}^{(D,F)}dt, \\ (c^{FX})^2 &= \int_0^T \sigma^{FX}(t)^2 dt, \\ c_i^{(D,FX)} &= \int_0^T \sigma_i^D(t)\sigma^{FX}(t)\rho_i^{(D,FX)}dt, \\ c_j^{(F,FX)} &= \int_0^T \sigma_j^F(t)\sigma^{FX}(t)\rho_j^{(F,FX)}dt. \end{aligned} \quad (6.17)$$

Then one has

$$\sigma_1^2 T = (c^{FX})^2 + 2\cdot\sum_{j=1}^{n}\{w_j^D(0)c_j^{(D,FX)} - w_j^F(0)c_j^{(F,FX)}\} + \sum_{j,k=1}^{n} \begin{Bmatrix} w_j^D(0)w_k^D(0)c_{jk}^{(D,D)} + w_j^F(0)w_k^F(0)c_{jk}^{(F,F)} \\ -2w_j^D(0)w_k^F(0)c_{jk}^{(D,F)} \end{Bmatrix}. \quad (6.18)$$

As a result, letting $\varepsilon = 1$, it follows that the first order normal asymptotic expansion of the FX call option price (6.2) is given by

$$C_{FX}^{(1)}(0) = P^D(0,T)\cdot\begin{bmatrix} (S(0)-K)\cdot\Phi\left(\frac{S(0)-K}{S(0)\sigma_1\sqrt{T}}\right) \\ +S(0)\sigma_1\sqrt{T}\cdot\varphi\left(-\frac{1}{2}\frac{S(0)-K}{S(0)\sigma_1\sqrt{T}}\right) \end{bmatrix} \quad (6.19)$$

where $\Phi(x)$ is the standard normal distribution and $\varphi(x) = \Phi'(x)$. Since the dynamics of the forex forward rate is close to a log-normal distribution it is very natural to approximate further the first order normal asymptotic expansion by the following log-normal distribution

$$F_T^{(\varepsilon)}(T) \sim \ln N\left(S(0) - \tfrac{1}{2}\sigma_1^2 T, \sigma_1\sqrt{T}\right), \quad (6.20)$$

$$C_{FX}^{BS(1)}(0) = P^D(0,T)\cdot\{S(0)\cdot\Phi\left(d_1^{(1)}\right) - K\cdot\Phi\left(d_2^{(1)}\right)\},$$
$$d_1^{(1)} = \frac{\ln\{S(0)/K\} + \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}, \quad d_1^{(2)} = d_1^{(1)} - \sigma_1\sqrt{T}. \quad (6.21)$$

Through differentiation one gets the first-order asymptotic expansion Black-Scholes delta

$$\frac{\partial C_{FX}^{BS(1)}(0)}{\partial S(0)} = P^D(0,T)\cdot\left\{\Phi\left(d_1^{(1)}\right) + \frac{S(0)\cdot\varphi\left(d_1^{(1)}\right) - K\cdot\varphi\left(d_2^{(1)}\right)}{S(0)\sigma_1\sqrt{T}}\right\}. \quad (6.22)$$

Second and third-order asymptotic expansion

Through similar but more complicated calculations it is possible to determine second and third order normal asymptotic expansions of the FX call option price as well as corresponding Black-Scholes approximations. Details of the third-order asymptotic expansion are found in Kawai and Jäckel [37], formulas (A.21) and (3.2).

To illustrate the use of these formulas and test their accuracy against MC simulation, we have calculated a single simple concrete example based on the input data of Tables 1 and 2.

Table 1: Characteristics of "FX option" contract

FX strike	0.900
FX rate	1.000
Time to maturity	4

Table 2: Market data

domestic/foreign forward rates & discount factors, forward FX strike & rate					
D forward rates		*D volatility*	*F forward rates*		*F volatility*
RD(0,1,2)	4.7598%	40%	RF(0,1,2)	4.7598%	40%
RD(0,2,3)	5.0489%	40%	RF(0,2,3)	5.0489%	40%
RD(0,3,4)	4.6725%	40%	RF(0,3,4)	4.6725%	40%
D discount factor			F discount factor		
PD(0,4)	0.830303		PF(0,4)	0.830303	
Forward strike	0.900	FX volatility			
Forward rate	1.000	10%			
correlation matrix					
D inter currency correlations			F inter currency correlations		
1	0	0	1	0	0
0	1	0	0	1	0
0	0	1	0	0	1
cross currency correlations			ρ(D,FX)	ρ(F,FX)	ρ(FX,FX)
0.5	0	0	0.2	-0.2	1
0	0.5	0	0.2	-0.2	
0	0	0.5	0.2	-0.2	

Numerical evaluation of the formulas (6.19), (6.21) and (6.22), as well as the corresponding third-order asymptotic expansion formulas in [37], is summarized in Table 3. For MC simulation of the forex forward rate according to (6.4) it is first necessary to calculate the spectral decomposition of the correlation matrix (4.10), here $C = B \cdot B^T$ as given in Table 4.

Table 3: Analytical asymptotic expansion formulas

Approximation formula	*Variance*	*Option Price*	*Option Delta*
1st term of 3rd order norm- al asymptotic expansion	0.01660	0.138	n.a.
1st order normal asymptotic expansion	0.01544	0.135	n.a.
3rd order log-normal (Black-Scholes)	0.01660	0.129	0.628
1st order log-normal (Black-Scholes)	0.01544	0.126	0.634

Table 4: Spectral decomposition (matrix B)

$\frac{\sqrt{3}}{2}$	0	0	$-\frac{1}{2}\sqrt{\frac{13}{55}}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{6}}{12}$	$-\frac{2}{\sqrt{165}}$
0	$\frac{\sqrt{3}}{2}$	0	$-\frac{1}{2}\sqrt{\frac{13}{55}}$	$-\frac{\sqrt{2}}{4}$	$\frac{\sqrt{6}}{12}$	$-\frac{2}{\sqrt{165}}$
0	0	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}\sqrt{\frac{13}{55}}$	0	$-\frac{\sqrt{6}}{6}$	$-\frac{2}{\sqrt{165}}$
$\frac{\sqrt{3}}{2}$	0	0	$\frac{1}{2}\sqrt{\frac{13}{55}}$	$-\frac{\sqrt{2}}{4}$	$-\frac{\sqrt{6}}{6}$	$\frac{2}{\sqrt{165}}$
0	$\frac{\sqrt{3}}{2}$	0	$\frac{1}{2}\sqrt{\frac{13}{55}}$	$\frac{\sqrt{2}}{4}$	$-\frac{\sqrt{6}}{6}$	$\frac{2}{\sqrt{165}}$
0	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}\sqrt{\frac{13}{55}}$	0	$\frac{\sqrt{6}}{12}$	$\frac{2}{\sqrt{165}}$
0	0	0	$-2\sqrt{\frac{13}{55}}$	0	0	1

For comparison we have computed MC simulations with two different Euler schemes, namely with the first usual one

$$F_T((i+1)h) = F_T(ih) \cdot \exp\left\{ \begin{array}{l} -\frac{1}{2}\left\|\gamma^{FX}(ih)+\eta(ih)\right\|^2 \cdot h \\ +\sqrt{h}\cdot\left(\gamma^{FX}(ih)+\eta(ih)\right)\cdot Z(ih) \end{array} \right\}, \quad i = 0,1,2,... \tag{6.23}$$

where

$$\begin{aligned} \left\|\gamma^{FX}(t)+\eta(t)\right\|^2 &= (\sigma^{FX})^2 \\ &+ 2\cdot\sum_{j=1}^{3}\left(\begin{array}{l} w_j^D(t)\sigma_j^D\sigma^{FX}\rho_j^{(D,FX)} \\ -w_j^F(t)\sigma_j^F\sigma^{FX}\rho_j^{(F,FX)} \end{array} \right) \\ &+ \sum_{j,k=1}^{3}\left(\begin{array}{l} w_j^D(t)w_k^D(t)\sigma_j^D\sigma_k^D\rho_{jk}^{(D,D)} \\ +w_j^F(t)w_k^F(t)\sigma_j^F\sigma_k^F\rho_{jk}^{(F,F)} \\ -2w_j^D(t)w_k^F(t)\sigma_j^D\sigma_k^F\rho_{jk}^{(D,F)} \end{array} \right) \end{aligned} \tag{6.24}$$

$$\begin{aligned} &\left(\gamma^{FX}(t)+\eta(t)\right)\cdot Z(t) = \sigma^{FX}W^{FX}(t) \\ &+\sum_{j=1}^{3} w_j^D(t)\sigma_j^D W_j^D(t) - \sum_{j=1}^{3} w_j^F(t)\sigma_j^F W_j^F(t) \\ &W^{FX}(t) = \sum_{i=1}^{7} b_{7i}Z_i(t), \quad W_j^D(t) = \sum_{i=1}^{7} b_{ji}Z_i(t), \\ &W_j^F(t) = \sum_{i=1}^{7} b_{3+ji}Z_i(t), \quad j = 1,2,3. \end{aligned} \tag{6.25}$$

The second Euler scheme is a simpler approximation of the first one, which is obtained by applying forward-drift approximations such that

$$F_T((i+1)h) = F_T(ih) \cdot \exp\left\{ \begin{array}{l} -\frac{1}{2}\left\|\gamma^{FX}(0)+\eta(0)\right\|^2 \cdot h \\ +\sqrt{h}\cdot\left(\gamma^{FX}(0)+\eta(0)\right)\cdot Z(ih) \end{array} \right\}, \quad i = 0,1,2,... \tag{6.26}$$

where $\left\|\gamma^{FX}(0)+\eta(0)\right\|^2 = \sigma_1^2$ coincides under the assumption of constant volatilities with (6.15). The discretization step is $h = 0.1$ and deltas are obtained with the path-wise finite difference or PFD method with a parallel shift $\varepsilon = 0.01$. The number $M = 10'000$ of simulated paths consists of 100 batches of 100 simulation paths each. Results are summarized in Table 5.

Table 5: Simulation of price and delta

	1st Euler scheme		2nd Euler scheme	
	Price	Delta	Price	Delta
MC FX values at time 0	0.123	0.611	0.120	0.612
1st order Black-Scholes approx.	0.126	0.634	0.126	0.634
MC relative deviations	3.1%	3.6%	4.8%	3.5%
3rd order Black-Scholes approx.	0.129	0.628	0.129	0.628
MC relative deviations	5.0%	2.6%	6.8%	2.5%

The option prices from the 1st order Black-Scholes approximation are closer to the MC values than those from the 3rd order Black-Scholes approximation. The opposite is true for the delta values. Moreover, the first Euler scheme yields closer approximations of the option prices to the MC values than the second one. But, no significant difference between the Euler schemes is observed concerning delta.

7. Conclusions and Outlook

Although a quite general LIBOR market model has been presented, a lot of issues remain open for investigation. Detailed implementation issues, further applications and careful testing are left to the practitioner. In the following, some more theoretical concerns are raised.

Clearly, the Gaussian setting of the general LIBOR market model can be questioned and there is an increased demand to go beyond it. However, the construction of multivariate non-Gaussian processes for financial purposes is a complex topic. For market consistent valuation one needs to deflate these processes with state price deflators. A possible approach to circumvent complexity can be based on normal variance-mean mixtures, as presented in author [30]. Here, we would like to emphasize modeling within the framework of copulas that are increasingly used in all areas of mathematical sciences (e.g. Cherubini et al. [8] for financial applications). Two main issues are discussed.

Compatibility of Correlation Matrices

The inaccurate approximation $\hat{C}$ of Example 2.2 suffers from another drawback within the copula setting. It is a simple example of a positive semi-definite correlation matrix which is not the rank correlation matrix of a multivariate normal copula. In fact, general compatibility conditions under which a joint normal distribution with a specified rank correlation matrix can be realized have only been derived quite recently in author [26], Corollary 4.2. One can ask for non-normal trivariate distributions for which $\hat{C}$ is compatible. From [26], Section 2, it is known that $\hat{C}$ is not compatible with a Bernoulli mixture trivariate reduction model. Note that the compatibility of three bivariate distributions in Fréchet spaces has been studied since Dall'Aglio [11], [12] (see also Rüschendorf [56], [57], Joe [33], Section 3, and Durante et al. [15]). The compatibility problem leads to our next issue.

Existence and Construction of Universal Copulas

In general, one says that a n-dimensional copula is n-universal if every n-dimensional valid correlation matrix can be realized as a rank correlation matrix, i.e. there exists a n-variate uniform distribution with this rank correlation structure. Clearly, there exist quite a lot of 2-universal copulas, the so-called comprehensive or inclusive copulas (see e.g. Nelsen [43]). Although the existence of 3-universal copulas has been settled by several authors (e.g. Joe [33], Exercise 4.17, pp. 137-138, Kurowicka and Cooke [38], Section 4.4.6, p.102, Devroye and Letac [13]), the effective construction of 3-universal copulas is more difficult. The author obtains in [31] an analytical 3-universal copula that is based on the bivariate linear circular copula in Perlman and Wellner [46]. The latter copula seems to have been independently obtained by Kurowicka et al. [39] under the naming „elliptical copula". According to Letac [40], the linear circular copula is a special case of probability distributions studied by Gasper [19]. It is remarkable that this copula can be used to settle the existence question for rank two extremal correlation matrices. By Theorem 7.1 below it generalizes to the copula framework the two-factor LMM of Rebonato [53] (see the Remarks 3.1). The construction, which mimics essentially Letac [40], is elementary.

Let $B_2 \subset R^2$ be the unit disk and $C_2 = [-1,1]^2$ the centred square. Consider the linear circular copula density with uniform [-1,1] margins U, V defined by

$$p_{(U,V)}(u,v) = \begin{cases} \dfrac{1}{2\pi\sqrt{1-u^2-v^2}}, & (u,v) \in B_2, \\ 0, & (u,v) \in C_2 - B_2. \end{cases} \quad (7.1)$$

Lemma 7.1 (*n-universal rank two extreme linear circular copula*) Given is the extreme correlation matrix of rank two of the form

$$r = (r_{ij}) = (\cos(\alpha_i - \alpha_j)), \quad \alpha_i \in [0,2\pi], 1 \le i, j \le n.$$

Then, there exist a random vector $(X_1, X_2, ..., X_n)$ with uniform [-1,1] margins $X_i, i = 1,...,n$, and rank two correlation matrix $r = (r_{ij})$.

Proof. Consider the random vector $(X_1, X_2, ..., X_n)$ defined by

$$X_i = \cos(\alpha_i) \cdot U + \sin(\alpha_i) \cdot V, \quad i = 1,...,n,$$

where the random pair (U, V) has the linear circular copula density (7.1). Clearly, the variables $X_i, i = 1,...,n$, are uniform [-1,1] random variables. Moreover, through application of the Jacobian transformation method, one sees that the probability density of $(X_i, X_j), 1 \le i \le j \le n$, is given by

$$p_{(X_i,X_j)}(x,y)=\begin{cases}\dfrac{1}{2\pi\sqrt{(1-r_{ij}^2)(1-x^2)-(y-r_{ij}x)^2}}, & (x,y)\in E_{r_{ij}}, \\ 0, & (u,v)\in C_2-E_{r_{ij}},\end{cases} \quad (7.2)$$

where the support $E_{r_{ij}}=\{(x,y)\mid x^2+y^2-2r_{ij}xy<1-r_{ij}^2\}$ is the inner of an ellipse, and $r_{ij}=\cos(\alpha_i-\alpha_j)$ coincides with the correlation coefficient of the pair (X_i,X_j) (e.g. Kurowicka et al. [39], Perlman and Wellner [46], author [31], Section 3). ◊

Theorem 7.1 (n*-universal rank two copula*) Given a rank two correlation matrix $r=(r_{ij}), 1\le i,j\le n$, there exist a random vector $(X_1,X_2,...,X_n)$ with uniform [-1,1] margins $X_i, i=1,...,n$, and rank two correlation matrix $r=(r_{ij})$.

Proof. This follows through application of the theorem of Carathéodory [7] and Steinitz [61]. Any valid correlation matrix (of rank two) is a finite convex combination of extreme correlation matrices (of rank two). Since the result holds for the extreme correlation matrices of rank two by Lemma 7.1, the result follows. ◊

Besides the existence of a single currency two-factor LMM of arbitrary dimension with arbitrary margins, Theorem 7.1 also settles the existence question for n-universal copulas, $n=3,4,5$. This follows because correlation matrices of dimensions $n=3,4,5$ have maximum rank two (e.g. Ycart (1985), Theorem 2).

Clearly, for simulation and analytical purposes, there is a need for alternative more explicit results. For the trivariate case, one can use the analytical results in [31], Theorem 5.3 and Theorem 6.1. Finally, a computational approach to (almost) n-universal copulas, which is based on doubly stochastic matrices and the checkerboard copula, has been proposed in the Sections 4 of Piantadosi et al. [47], [48]. Future researchers should appreciate the challenge to extend classical LIBOR market models to a copula setting.

References

[1] H. Abdi, L.J. Williams, "Principal component analysis," WIREs Computational Statistics 2(4), pp. 433-459, 2010.
[2] C. Alexander, D. Lvov, "Statistical properties of LIBOR forward rates," ISMA Discussion Papers, 2003. [Online]. Available: http://www.icmacentre.ac.uk/pdf/discussion/DP2003-03.pdf. [Accessed: Nov. 11, 2014].
[3] L. Andersen, J. Andreasen, "Factor dependence of Bermudan swaptions: fact or fiction?" Journal of Financial Economics 62(1), pp. 3-37, 2001.
[4] F. Black, "The pricing of commodity contracts," J. of Financial Economics 3, pp. 167-179, 1976.
[5] D. Brigo, "A note on correlation and rank reduction," 2002. [Online]. Available: www.damianobrigo.it/correl.pdf. [Accessed: Nov. 11, 2014].
[6] M. Budden, P. Hadavas, L. Hoffman, "On the generation of correlation matrices," Applied Mathematics E-Notes 8, pp. 279-282, 2008.
[7] C. Carathéodory, "Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen," Rend. del Circolo Matem. Palermo 32, pp. 193-217, 1911.
[8] U. Cherubini, E. Luciano, W. Vecchiato, Copula Methods in Finance, J. Wiley Finance Series, New York, 2004.
[9] M. Chesney, L. Scott, "Pricing European currency options: a comparison of the modified Black-Scholes model and a random variance model," J. Financial and Quantitative Analysis 24, pp. 267-284, 1989.
[10] Q. Dai, K. Singleton "Term structure dynamics in theory and reality," Review of Financial Studies 16(3), pp. 631-678, 2003.
[11] G. Dall'Aglio "Sulla compatibiltà delle funzioni di repartizione doppia," Rend. Mat. 18, pp. 385-413, 1959.
[12] G. Dall'Aglio, "Fréchet classes and compatibility of distribution functions," In: Symposia Mathematica, vol. IX, pp. 131-150, 1972, Academic Press, London.
[13] L. Devroye, G. Letac, "Copulas in three dimensions with prescribed correlations," 2010. [Online]. Available: http://arxiv.org/abs/1004.3146v1 [math.ST]. [Accessed: Nov. 11, 2014].
[14] J. Driessen, P. Klaassen, B. Melenberg, "The performance of multi-factor term stucture models for pricing and hedging caps and swaptions," J. of Financial and Quantitative Analysis 38(3), pp. 635-672, 2003.
[15] F. Durante, E.P. Klement, J.J. Quesada-Molina, "Copulas: compatibility and Fréchet classes," Preprint, 2007. [Online]. Available: arXiv:0711.2409v1 [math.ST]. [Accessed: Nov. 11, 2014].
[16] R. Fan, A. Gupta, P. Ritchken, "Hedging in the possible presence of unspanned stochastic volatility: evidence from swaption markets," J. of Finance 58(5), pp. 2219-2248, 2003.
[17] Ch. Fries, Mathematical Finance - Theory, Modeling and Implementation, J. Wiley, New York, 2007.
[18] B. Flury, Common principal components and related multivariate models, J. Wiley, 1988.
[19] G. Gasper, "Banach algebra for Jacobi series and positivity of a kernel," Annals of Mathematics 95, pp. 261-280, 1971.
[20] P. Glasserman, X. Zhao, "Fast Greeks by simulation in forward Libor models,"Journal of Computational Finance 3, pp. 5–39, 1999.
[21] I. Grubisic, R. Pietersz, "Efficient rank reduction of correlation matrices," Linear Algebra and its Applications 422, pp. 629-653, 2007.
[22] A. Gupta, M. Subrahmanyam, "Pricing and hedging interest rate options: evidence from cap-floor markets," J. of Banking and Finance 29(3), pp. 701-733, 2005.
[23] S.L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and

currency options," Review of Financial Studies 6, pp. 327-343, 1993.
[24] N.J. Higham, "Computing the nearest correlation matrix – a problem from finance," IMA Journal of Numerical Analysis 22, pp. 329-344, 2002.
[25] W. Hürlimann, "An extension of the Black-Scholes and Margrabe formulas to a multiple risk economy," Applied Mathematics 2(4), pp. 137-142, 2011.
[26] W. Hürlimann, "Compatibility conditions for the multivariate normal copula with given rank correlation matrix," Pioneer Journal of Theoretical and Applied Statistics 3(2) (2012), pp. 71-86, 2012.
[27] W. Hürlimann, "Option pricing in the multidimensional Black-Scholes market with Vasicek interest rates. Math. Finance Lett. 2(1) (2013), pp. 1-18, 2013.
[28] W. Hürlimann, "Option pricing in the multidimensional Black-Scholes-Merton market with Gaussian Heath-Jarrow-Morton interest rates: the parsimonious and consistent Hull-White models of Vasicek and Nelson-Siegel type," Math. Finance Lett., Vol. 2014, Article ID 3, 2014.
[29] W. Hürlimann, "Cartesian and polar coodinates for the n-dimensional elliptope," Theoretical Mathematics & Applications 4(3), 1-17, 2014.
[30] W. Hürlimann, "Normal variance-mean mixtures (III) Option pricing through state-price deflators," J. of Mathematical Sciences: Advances and Applications 25(1), pp. 13-41, 2014.
[31] W. Hürlimann, "A closed-form universal trivariate pair-copula," Journal of Statistical Distributions and Applications 1(7), 25p., 2014.
[32] P. Jäckel, C. Kahl, "Positive semi-definite correlation marix completion," 2009. [Online]. Available: http://www.pjaeckel.webspace.virginmedia.com/PositiveSemidefiniteCorrelationMatrixCompletion.pdf. [Accessed: Nov. 11, 2014].
[33] H. Joe, Multivariate models and dependence concepts, Monographs on Statistics and Applied Probability, vol. 73, Chapman & Hall, London, 1997.
[34] I.T. Joliffe, Principal Component Analysis (2nd ed.), Springer-Verlag, New York, 2002.
[35] C. Kahl, "Modeling and simulation of stochastic volatility in finance," PhD Thesis, Bergische Universität Wuppertal and ABN Amro, 2007.
[36] C. Kahl, M. Günther "Complete the correlation matrix," 2005. [Online]. Available: http://www2.math.uni-wuppertal.de/~kahl/publications/CompleteTheCorrelationMatrix.pdf. [Accessed: Nov. 11, 2014].
[37] A. Kawai, P. Jäckel, "An asymptotic FX option formula in the cross currency LIBOR market model," Willmott, March 2007, 74-84, March, 2007.
[38] D. Kurowicka, M.R. Cooke, Uncertainty analysis with high dimensional dependence modelling, J. Wiley, Chichester, England, 2006.
[39] D. Kurowicka, D., M.R.Cooke, J. Misiewicz, "Elliptical copulae," In: GI Schuëller & PD Spanos (Eds.), Proceedings Monte Carlo Simulation, pp. 209-214, Lisse: Balkema, 2000.
[40] G. Letac, "The convex set of correlation matrices and Jacobi polynomials," Conference Presentation, Orthogonal Polynomials and Hypergroups, Toulouse, June 18, 2014.
[41] M. Lim, "Calibrating a 3-factor LIBOR market model," Ecole Polytechnique, 2004. [Online]. Available: http://homepages.nyu.edu/~mrl306/fichiers/rapporte.pdf [Accessed: Nov. 11, 2014].
[42] F. Longstaff, P. Santa-Clara, E. Schwartz, "Throwing away a billion dollars: the cost of suboptimal exercise strategies in the swaptions market," Journal of Financial Economics 62(1), pp. 39-66, 2001.
[43] R.B. Nelsen, An introduction to copulas, Lecture Notes in Statistics, vol. 139 (2nd edition). Springer-Verlag, New York, 2006.
[44] P.J. Olver, "Orthogonal bases and the QR algorithm" University of Minnesota, 2008. [Online]. Available: http://www.math.umn.edu/~olver/aims_/qr.pdf. [Accessed: Nov. 11, 2014].
[45] N. Packham, "Correlation parameterization and calibration for the LIBOR market Model," MsC Thesis, BS Finance & Management, Frankfurt am Main, 2005.
[46] M.D. Perlman, J.A. Wellner, "Squaring the circle and cubing the sphere: circular and spherical copulas," Symmetry 3, pp. 574-599, 2011.
[47] J. Piantadosi, P.G. Howlett, J.W. Boland, "Matching the grade correlation coefficient using a copula with maximum disorder," J. Indust. Man. Opt. 3(2), pp. 305-312, 2007.
[48] J. Piantadosi, P.G. Howlett, J.W. Borwein, "Copulas with maximum entropy," Optim. Lett. 6, pp. 99-125, 2012.
[49] R. Pietersz, P. Groenen, "Rank reduction of correlation matrices by majorization", Quantitative Finance 4(6), pp. 649-662, 2004.
[50] R. Pietersz, A. Pelsser, "A comparison of single Markov-functional and multifactor market models," Review of Derivatives Research 13, pp. 245-272, 2010.
[51] F. Rapisarda, D. Brigo, F. Mercurio, "Parameterizing correlations: a geometric interpretation," IMA Journal of Management Mathematics 18(1), pp. 55-73, 2007.
[52] R. Rebonato, Interest-rate Option Models, J. Wiley, New York, 1996.
[53] R. Rebonato, "On the simultaneous calibration of multifactor lognormal interest rate models to Black volatilities and to the correlation matrix," J. Comp. Finance 2, pp. 5-27, 1999.
[54] R. Rebonato, Volatility and Correlation (2nd ed.), J. Wiley, New York, 2004.
[55] R. Rebonato, P. Jäckel, "The most general methodology to create a valid correlation matrix for risk management and option pricing purposes," Journal of Risk 2(2), pp. 17-27, 2000.
[56] L. Rüschendorf, "Bounds for distribution with multivariate marginal," In: Stochastic Orders and Decision under Risk, IMS Lecture Notes Monograph Series, vol. 19, Institute Math. Statist., Hayward, 1991.
[57] L. Rüschendorf, "Fréchet bounds and their applications," In: G. Dall'Aglio, S. Kotz, G. Salinetti (Eds.). Advances in Probability Distributions with Given Marginals, Kluwer, Dordrecht, 1991.
[58] R. Schöbel, J. Zhu, "Stochastic volatility with an Ornstein Uhlenbeck process: an Extension," European Finance Review 3, pp. 23-46, 1999.

[59] J. Schoenmakers, C. Coffey, "LIBOR rate models, related derivatives and calibration," WIAS Preprint No. 480, Weierstrass Institute Berlin, 1999.
[60] J. Schoenmakers, C. Coffey, "Systematic generation of parametric correlation structures for the LIBOR market model," Int. J. Theor. Applied Finance 6(5), pp. 507-519, 2003.
[61] E. Steinitz, "Bedingt konvergente Reihen und konvexe Systeme," Journal Reine Angew. Math. 143, pp. 128–175, 1914.
[62] L. Tamminen, "Pricing of constant maturity spread options in the deterministic LIBOR market model framework," MsC Thesis, Tampere University of Technology, Finnland, 2008.
[63] B. Ycart, "Extreme points in convex sets of symmetric matrices," Proceedings of the American Mathematical Society 95(4), pp. 607-612, 1985.
[64] G. Yule, M. Kendall, An Introduction to the Theory of Statistics (14th ed.), Charles Griffin & Company, Belmont, California, 1965.
[65] Z. Zhang, L. Wu, "Optimal low-rank approximation to a correlation matrix," Linear Algebra and its Applications 364, pp. 161-187, 2003.

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