On the Lattice of L-closure Operators

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Abstract: In this paper we proved that the lattice of all L-closure operators on a fixed set X is not modular. We identified the infra L-closure operators and ultra L-closure operators. Also established the relation between ultra L-topologies and ultra L-closure operators.

Keywords: L - closure operator, ultra L-topology, ultra L-closure operator, infra L-closure operator.

1. Introduction

In 1965 L. A. Zadeh [11] introduced fuzzy sets as a generalization of ordinary sets. After that C. L. Chang [2] introduced fuzzy topology and that led to the discussion of various aspects of L-topology by many authors. The Čech closure spaces introduced by Čech E. [1] is a generalization of the topological spaces. The theory of fuzzy closure spaces has been established by Mashhour and Ghanim [4] and Srivastava et. al [6],[7]. The definition of Mashhour and Ghanim is an analogue of Čech closure spaces and Srivastava et. al. have introduced it as an analogue of Birkhoff closure spaces in [7]. Based on [7], Rekha Srivastava and Manjari Srivastava studied the subspace of a fuzzy closure space. The notion of T₀-fuzzy closure spaces and T₁ fuzzy closure spaces were also introduced in [6]. In [5] P. T. Ramachandran studied the properties of lattice of closure operators. In [3] T. P. Johnson studied some properties of the lattice L(X) of all fuzzy closure operators on a fixed set X. In [9] Wu-Neng Zhou introduced the concept of L-closure operator as follows.

2. Preliminaries

A completely distributive lattice L is called a Fuzzy lattice, if there is an order reversing involution from L to L. Let X be any nonempty set and L is a Fuzzy lattice. The fundamental definition of L-fuzzy set theory and L-fuzzy topology are assumed to be familiar to the reader as in [10].Here we call L-fuzzy subsets as L subsets and L-fuzzy topology as L-topology.

2.1 Definition

A Čech fuzzy closure operator on a set X is a function χ : ĆX → ĆX , satisfying the following three axioms
1. χ(0) = 0
2. f ≤ χ(f) for every f in ĆX.
3. χ(f ∨ g) = χ(f) ∨ χ(g) where I = [0, 1]

For convenience it is called fuzzy closure operator on X and (X, χ) is called fuzzy closure space. In [9] Wu-Neng Zhou defined L-closure operator as follows.

2.2. Definition

A mapping C : L X → L X is called an L-closure operator or an L-closure, if it satisfies the following conditions for any A, B ∈ L X :
1. C(0X ) = 0X
2. A ≤ C(A)
3. A ≤ B implies C(A) ≤ C(B)
4. C(C(A)) = C(A)

But in this paper we take the definition of L-closure operator as a generalization of fuzzy closure operator in [4]

2.3. Definition

Let X be a nonempty set and L be a Fuzzy lattice. An L-closure operator on L X is a mapping ψ : L X → L X satisfying the following conditions:
1. ψ(0) = 0
2. f ≤ ψ(f)
3. ψ(f ∨ g) = ψ(f) ∨ ψ(g) for every f, g ∈ L X

The pair (X, ψ) is called an L-closure space. An L-subset f of X is said to be an L-closed set in (X, ψ) if ψ(f) = f. An L subset f of X is open if its complement is
closed in \((X, \psi)\). The set of all open \(L\) subsets of \((X, \psi)\) form an \(L\)-topology on \(X\) called the \(L\)-topology associated with the \(L\)-closure operator \(\psi\).

Let \(F\) be an \(L\)-topology on a set \(X\). Then a function \(\psi : L^X \rightarrow L^X\) defined by \(\psi(f) = \overline{f}\) for all \(f \in L^X\), where \(\overline{f}\) denotes the closure of \(f\) with respect to \(F\) is called the \(L\)-closure operator associated with the \(L\)-topology \(F\).

An \(L\)-closure operator on a set \(X\) is called \(L\)-topological if it is the \(L\)-closure operator associated with an \(L\)-topology on \(X\). That is \(\psi(\psi(f)) = \psi(f)\) for all \(f \in L^X\)

Note that different \(L\)-closure operators can have the same associated \(L\)-topology. But different \(L\)-topologies cannot have the same associated \(L\)-closure operator.

3. Lattice of \(L\)-closure operators

Let \(\psi_1\) and \(\psi_2\) be \(L\)-closure operators on \(X\). Then \(\psi_1 \leq \psi_2\) if and only if for every \(f \in L^X\), \(\psi_1(f) \subseteq \psi_2(f)\). The relation \(\leq\) defined above is a partial order on the set of all \(L\)-closure operators on \(L^X\). We denote the poset by \(LC(X)\). Then \(LC(X)\) is a lattice. The \(L\)-closure operator \(D\) on \(X\) defined by \(D(f) = f\) for every \(f \in L^X\) is called the discrete \(L\)-closure operator. The \(L\)-closure operator \(I\) on \(X\) defined by \(I(f) = 0\) if \(f = 0\) and \(I(f) = 1\) otherwise is called the indiscrete \(L\)-closure operator.

**Remark 3.1**

D and I are the \(L\)-closure operators associated with the discrete and indiscrete \(L\)-topologies on \(X\) respectively. Moreover \(D\) is the unique \(L\)-closure operator who’s associated \(L\)-topology is discrete. Also I and \(D\) are the smallest and the largest elements of \(LC(X)\) respectively.

**Theorem 3.1.**

\(LC(X)\) is a complete lattice.

**Proof.** Can be easily proved.

**Definition 3.1.**

Lattice of \(L\) closure operators \(LC(X)\) is modular if and only if

\[\chi \leq \eta \implies \chi \land (\psi \lor \eta) = (\chi \land \psi) \lor \eta \lor (\chi \lor \psi) \lor \eta \in LC(X)\]

**Theorem 3.2.**

\(LC(X)\) is not modular

**Proof.** Let \(X\) be any set and \(x \in X\). Define \(\psi_x, \chi_x, \eta_x\) from \(L^X\) to \(L^X\) by \(\psi_x(0) = 0\), \(\psi_x(f)(y) = \begin{cases} 1 & \text{if } y = x \\ \overline{f(y)} & \text{otherwise} \end{cases}\), \(\chi_x(0) = 0\), \(\chi_x(f)(y) = \begin{cases} 1 & \text{if } y = x \\ \overline{f(y)} & \text{otherwise} \end{cases}\), \(\eta_x(0) = 0\), \(\eta_x(f)(y) = \begin{cases} 1 & \text{if } y = x \\ \overline{f(y)} & \text{otherwise} \end{cases}\). Let \(\varphi\) be any \(L\) closure operator other than \(I\). Then we can find a nonzero \(L\) subset \(f\) such that \(\varphi(f) \neq I\).

**Definition 3.2.**

An \(L\)-closure operator on \(X\) is called an infra \(L\)-closure operator if the only \(L\)-closure operator on \(X\) strictly smaller than \(\psi\) is \(0\). According to the following theorem.

**Theorem 3.3.**

An \(L\)-closure operator on \(X\) is called an infra \(L\)-closure operator if and only if it is of the form \(\varphi \land \psi\), for some \(\varphi\), \(\psi\) \(\in LC(X)\), \(\varphi \neq 0\).
(f) = \{ f \} (i.e. \psi(f) \neq \emptyset ) and elements \( a_k, b_k \) where \( a, b \in L \) such that \( a_k \leq f \) and \( b_k \) not in \( \psi(f) \). Then \( \psi(f) \) is not an element of \( \psi(a_k) \). That is \( b_k \neq \emptyset \) \( \Rightarrow \psi(a_k) \neq \emptyset \). That is \( \forall a, b (a_k) \leq \psi(a_k) \). Also \( \forall a, b (k) = 1 \) for every nonzero L subset \( k \) other than \( a_k \). Let \( \forall a, b (f) \geq \psi(f) \) \( \forall f \). That is \( \forall a, b (f) \leq \psi \). Thus all infra L-closure operators are of the form \( \forall a, b \) for \( a, b \in X \) such that \( a \neq b \).

### 3.2 Remark

When \( L = I \) there is no infra L-closure operator.

#### Definition 3.3

An L-topology \( F \) on \( X \) is an ultra L-topology if the only L-topology on \( X \) strictly finer than \( F \) is the discrete L-topology.

Let \( X \) be a nonempty set and \( L \) is a finite pseudo complemented chain. If \( \overline{\omega} = \overline{\omega}(a, \overline{\omega}(b)) = \{ f \mid f(a) = 0 \} \cup \{ f \mid f \geq b \} \), then a principal ultra L-topology is \( \overline{\omega}(a, \overline{\omega}(b)) = \overline{\omega}(a) \), which is the simple extension of \( \overline{\omega} \) by \( a \), i.e. \( \overline{\omega}(a) = \{ f \mid f \geq a \} \), where \( a, b \in X, \lambda \) and \( \beta \) are the atom and dual atom in \( L \) respectively.

Let \( X \) be a nonempty set and \( L \) is a Boolean lattice. If \( \overline{\omega} = \overline{\omega}(a, \overline{\omega}(b)) = \{ f \mid f(a) = 0 \} \cup \{ f \mid f \geq b \} \), where \( a, b \in X, \lambda \) is an atom, then a principal ultra L-topology denoted by \( \overline{\omega}(\beta) \) = L-topology generated by any \( (m - 1) \), \( \overline{\omega}(a_k) \) among \( m \), \( \overline{\omega}(a_k) \), \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, m, \lambda \) and \( \lambda \) are the atom and dual atom in \( L \) respectively.

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Let \( X \) be an infinite set and \( L \) is a finite pseudo complemented chain.

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Let \( X \) be an infinite set and \( L \) is a Boolean lattice. If \( \overline{\omega} = \overline{\omega}(a, \overline{\omega}(b)) = \{ f \mid f(a) = 0 \} \cup \{ f \mid f \geq b \} \), where \( a, b \in X, \lambda \) is an atom, then a principal ultra L-topology denoted by \( \overline{\omega}(\beta) \) = L-topology generated by any \( (m - 1) \), \( \overline{\omega}(a_k) \) among \( m \), \( \overline{\omega}(a_k) \), \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, m, \lambda \) and \( \lambda \) are the atom and dual atom in \( L \) respectively.

**Theorem 3.4**

Let \( X \) is a non empty set and \( L \) is a diamond lattice \( \{0, a, b, 1\} \). Then an L-closure operator on \( X \) is ultra L-closure operator if and only if it L-closure operator associated with some ultra L-topology on \( X \).

**Proof.** Let \( \psi(a, \overline{\omega}(b)) \) be an ultra L-topology on \( X \) and \( \psi \) be the associated L-closure operator. Let \( \psi' \) be an L-closure operator on \( X \) strictly larger than \( \psi \). Then there exists an L subset \( f \) of \( X \) such that \( \psi'(f) \neq \psi(f) \).

**Remark 3.3**

In a similar way we can prove the above theorem when \( L \) is a finite pseudo complemented chain or other Boolean lattice.

**Definition 3.6**

Let \( \psi = \{ f | \psi(f) = f \} \). A fuzzy closure space \( (X, \psi) \) is called quasi-separated if and only if for any two fuzzy points \( x \) and \( y \) with \( x \in C(f) \), there exist \( f, g \in \psi \), such that \( x \in f \leq C(y) \) and \( y \in g \leq C(x) \).

**Theorem 3.5**

A fuzzy closure space is quasi-separated if and only if every fuzzy point in \( X \) is Čech-fuzzy closed.
Result

Let $\psi_1 = \{ f \in L^X \mid \psi(f) = f \}$. An L-closure space $(X, \psi)$ is said to be $T_1$ if for every pair of distinct L points $x_i$ and $y_i$, there exist $f_i, g_i \in \psi_1$ such that $x_i \in \psi f_i \leq C(y_i)$ and $y_i \in g_i \leq C(x_i)$.

**Proof.** Necessary part

Suppose that the L-closure operator $\psi$ is $T_1$. Then by definition $\psi (x_i) = x_i$. Then by theorem 3.5 the L-closure space $(X, \psi)$ is quasi separated. Hence for every pair of distinct L points $x_i$ and $y_i$, there exist $f, g \in \psi_1$ such that $x_i \in \psi f \leq C(y_i)$ and $y_i \in g \leq C(x_i)$. Sufficient part Suppose that for every pair of distinct L points $x_i$ and $y_i$, there exist $f, g \in \psi_1$ such that $x_i \in \psi f \leq C(y_i)$ and $y_i \in g \leq C(x_i)$. Then by definition $(X, \psi)$ is quasi separated. Then by theorem 3.5, $(X, \psi)$ is a $T_1$ L-closure space.

Proposition 8 An L-closure space $(X, \psi)$ is $T_1$ if and only if the associated L topological space $(X, F)$ is $T_1$.

Theorem 3.6.

Infra L-closure operators are less than or equal to any non principal ultra L-closure operator.

**Proof.** Let $\psi_{a,b}$ be an infra L-closure operator and $\psi$ be a non principal ultra L-closure operator. Since $\psi_{a,b} \psi_{a,b}(f) = 1$

Theorem 3.7

No non principal ultra L-closure operator has a complement.

**Proof.** Assume the contrary. Let $\psi$ be a non principal ultra L-closure operator with a complement $\psi'$ in the lattice LC(X). Since $\psi'$ is not indiscrete there exists an infra L-closure operator $\psi_{a,b} \leq \psi'$ by the proof of the theorem 3.3. But $\psi_{a,b} \leq \psi$ by theorem 3.6. This contradicts the fact that $\psi$ and $\psi'$ are complements in the lattice LC(X) and hence the proof of the theorem.

Remark 3.4.

The lattice of L-closure operators is not complemented in general.

If L is a diamond lattice, the principal ultra L-closure operator associated with the principal ultra L-topology $\delta(a, (b_i), a_o)$ is given by $\phi_{a,b}(f) = f$ if $f = 0$ or $a_o \leq f$ or $cf \in \psi (b_i)$ $f \vee a_o$ otherwise

Theorem 3.8.

An infra L-closure operator $\psi_{a,b}$ and $\phi_{b,a}$ are incomparable if L is a diamond lattice.

**Proof.** We have $\psi_{a,b}(a_o) = g_{c,\beta}, \phi_{b,a}(a_o) = a_o \vee b_o$. Since $\alpha$ and $\beta$ are not comparable, $\psi_{a,b}$ and $\phi_{b,a}$ are not comparable.

Remark 3.5.

In a similar way, we can discuss the above theorem if L is a finite pseudo complemented chain or other Boolean lattices.

4 Conclusion

In this paper we identified the infra L-closure operator and ultra L-closure in LC(X) and established the relation between ultra L-closure topology and ultra L-closure operator if there is a dual atom in the lattice L. Also it is proved that LC(X) is not modular and not complemented in general.

5 Future Scope

The problem of finding whether this lattice is atomic and dually atomic under any condition on the fuzzy lattice L, is not yet solved. Also, the problem of semi-modularity and semi-complementation is not yet analyzed.

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References


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Raji George received the B.Sc. and M.Sc. degrees in Mathematics from St. Peter’s College, Kolenchery in 1984 and 1986 respectively, B.Ed. from Govt. Trainings College, Trichur, and M.Phil degree from Madurai Kamaraj University. She is currently Associate Professor in St. Peter’s College, Kolenchery.