A Study on Achromatic Coloring Graphs and its Applications

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Abstract: The achromatic number $\chi_a (G)$ of a graph is the greatest number of color in a vertex coloring such that each pair of colors appears on at least one edge. In this paper we give some properties of achromatic coloring for complete graphs and its applications.

Keywords: Achromatic coloring, achromatic number, Lower bound, Upper bound

1. Introduction

1.1 Graph Coloring
It is more than 200 years old. It has a many practical applications like computer science, telecommunications, operation research, designs of experiments etc. It is a special case of graph labelling.

1.2 Achromatic Coloring
It is a proper vertex coloring such that each pair of color classes is adjacent by at least one edge. The largest possible number of colors in an achromatic coloring is called the achromatic number and is denoted by $\chi_a(G)$, where $G$ is a finite undirected graph with no loops and multiple edges.

The achromatic number was defined and studied by Harary, Hedetniemi and prins [2]. Yannakakis and Gavril [5] Proved that determining this value for general graphs is NP – complete. The NP-completeness of the achromatic number for trees was established by Cairnie and Edwards [6].

Further it is solvable for paths, cycles, complete bipartite graphs. Roichman [9] gives the achromatic number for Hypercubes.

2. Definitions

2.1 Chromatic Coloring and Chromatic Number
Let $G (V, E)$ be a graph and $\phi : V \rightarrow N$ a labelling of the vertices. $\phi$ is a proper coloring if for every $u,v \in V$ with $(u,v) \in E$ then $\phi (u) \neq \phi (v)$, in other words neighbouring vertices should not be assigned the same color. The chromatic number denoted by $\chi (G)$.

2.2. Achromatic Coloring and Achromatic Number
The achromatic coloring of a graph is a proper vertex coloring such that each pair of color classes is adjacent by at least one edge.

The largest possible number of colors in an achromatic coloring of a graph $G$ is called the and it is denoted by $\chi_a(G)$.

2.3 Pseudo Achromatic Number
The pseudo achromatic number $\alpha (G)$ is the maximum $k$ for which there exists a complete coloring of $G$. If the coloring is required also to be proper, then such a maximum is known as the achromatic number and it will be denoted here by $\chi_a(G)$.

2.4 Line Distinguishing Coloring
Let $G (V, E)$ be a graph. A coloring $\phi : V \rightarrow N$ of the vertices is a line distinguishing coloring iff for every edge $(u,v) \in E$ the edge color $(\phi (u), \phi (v))$ is unique, (i.e). It appears at most once.

2.5 Central graph
Let $G$ be a finite undirected graph with no loops and multiple edges. The central graph $C(G)$ of a graph is obtained by subdividing each edge of $G$ exactly once and joining all the non adjacent vertices of $G$.

Table 1: Properties of Achromatic Coloring
In general graphs it is difficult to find lower bound. For particular graphs with large girth (at least 5) admit algorithms with relatively low approximation ratio for the achromatic number. This result gives on the observation that $\chi_a(G) \leq m/n$ for graphs G with n vertices, m edges and girth at least 5.

**Upper Bound**

In general graphs it is difficult to find upper bound. For a particular case approximating the achromatic number for general or bipartite graphs, the approximation ratio guarantees are just barely sub-linear in the number of vertices.

**NP-complete**

The problem of achromatic coloring is NP-complete for general graphs.

**NP hard**

The problem of achromatic coloring is NP-hard for trees, bipartite graphs, interval graphs, bipartite permutation and quasi-threshold graphs.

Some of the most important results on achromatic coloring graphs, that appeared in the literature survey. For studying these particular kinds of coloring by presenting several potential applications [8].

### 3. Observation

**Theorem 3.1:**

The pseudo achromatic number $\alpha(G)$ is the maximum k for which there exists a complete coloring of G. If the coloring is required also to be proper, then such a maximum is known as the achromatic number and it will be denoted here by $\chi_a(G)$. Clearly $\chi(G) \leq \chi_a(G) \leq \alpha(G)$, where $\chi(G)$ denotes, the chromatic number of G.

**Theorem 3.2:**

Exhibiting an explicit coloring and showing a general upper bound is follows that, if $q = 2^\beta$, for some $\beta \in \mathbb{N}$, and $n = q^2 + q + 1$, then $q^3 + q \leq \alpha(n) \leq q^3 + \frac{1}{2} q^2 + 1$. Besides those implied by Bouchet’s theorem, very few exact values for $\chi_a(n)$ are known.

**Corollary 3.1.1:**

We were able to calculate exactly $\alpha(n)$ in the following family, if $q = 2^\beta$, for some $\beta \in \mathbb{N}$, and $n = q^2 + 2q + 2$, then $\alpha(n) = q(n+1) = q^3 + 2q^2 + 3q$.

Let G be a graph and H be a sub graph of G. Then $\chi_a(G) \geq \chi_a(H)$.

The following observation is from [4] for the complete graphs $C_n, K_n$ etc.

**Theorem 3.4**

For any complete graph $K_n$, the number of edges in $L[C(K_n)] = n(n-1)/2$.

**Theorem 3.5**

For any cycle $K_n$, $\chi_a(L[C(K_n)]) = 2n-3$.

**Theorem 3.6**

For any complete graph $K_n$, $\chi_a(L[C(K_n)]) = 2n-3$, Here $L[C(K_n)]$ satisfies the following properties

(i) Maximum degree of vertices = Minimum degree of the vertices = n-1.
(ii) $L[C(K_n)]$ contain n copies of vertex disjoint $K_{n-1}$.
(iii) There is a cycle ‘c’ of length 2n with alternate edges from each of the complete graph $K_{n-1}$.

In this paper, we improve the following corollary by the observation of theorem 3.6 with the example. Here the restriction is $n \geq 3$.

**Corollary 3.1.2**

For any complete graph $K_n$, $\chi_a(L[C(K_n)]) = 2n-3$, $n \geq 3$

Proof:

In theorem 3.6, Substitute

(i) $n=1$ then $\chi_a(L[C(K_n)]) = -1$ (- ve)
(ii) $n=2$ then $\chi_a(L[C(K_n)]) = 1$ (+ ve).

Here the graph is the complete graph, so that in this corollary our restriction is $n \geq 3$.

Following example is given for line graph of central graph of $K_3$ is equal to 9.

**Example 3.2.1**

Consider $L(C(K_3))$ with 6 copies of vertex disjoint $K_3$ complete graph which satisfies maximum and minimum degree of the vertices are 5 then $\chi_a(L[C(K_3)]) = 2(6) - 3 = 9$.
4. Applications

4.1 Graph theory

It has many applications like, radio navigation, and image compression. In computer science, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation, etc.

4.2 Graph coloring

The problem of coloring a graph has number of applications. Some of them are scheduling, bandwidth allocation, pattern matching and puzzle Sudoku. Few examples are given as follows:

(List coloring, Multi coloring, Minimum sums coloring, Harmonious Coloring etc).

4.3 Achromatic Coloring:

Greedy approach for finding an achromatic coloring with a large number of colors However, using a semi-greedy approach to extracting small independent sets, Chaudhary and Vishwanathan [1] gave the first sub linear approximation algorithm for the achromatic number problem with an approximation ratio of \(O(n^{\sqrt{\log n}})\) for any graph with \(n\) vertices. They conjectured that the achromatic number can be approximated with in a ratio of \(O(p\chi_a(G))\) for any graph \(G\). In support of their conjecture, they gave an algorithm that returns a \(O(p\chi_a(G)) = O(n^{7/20})\) ratio approximation for graphs \(G\) with girth (i.e. length of the shortest simple cycle) at least 7.

5. Conclusion

In this paper we discussed about the achromatic number of central graphs and its properties. This type of coloring gives several applications.

References


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